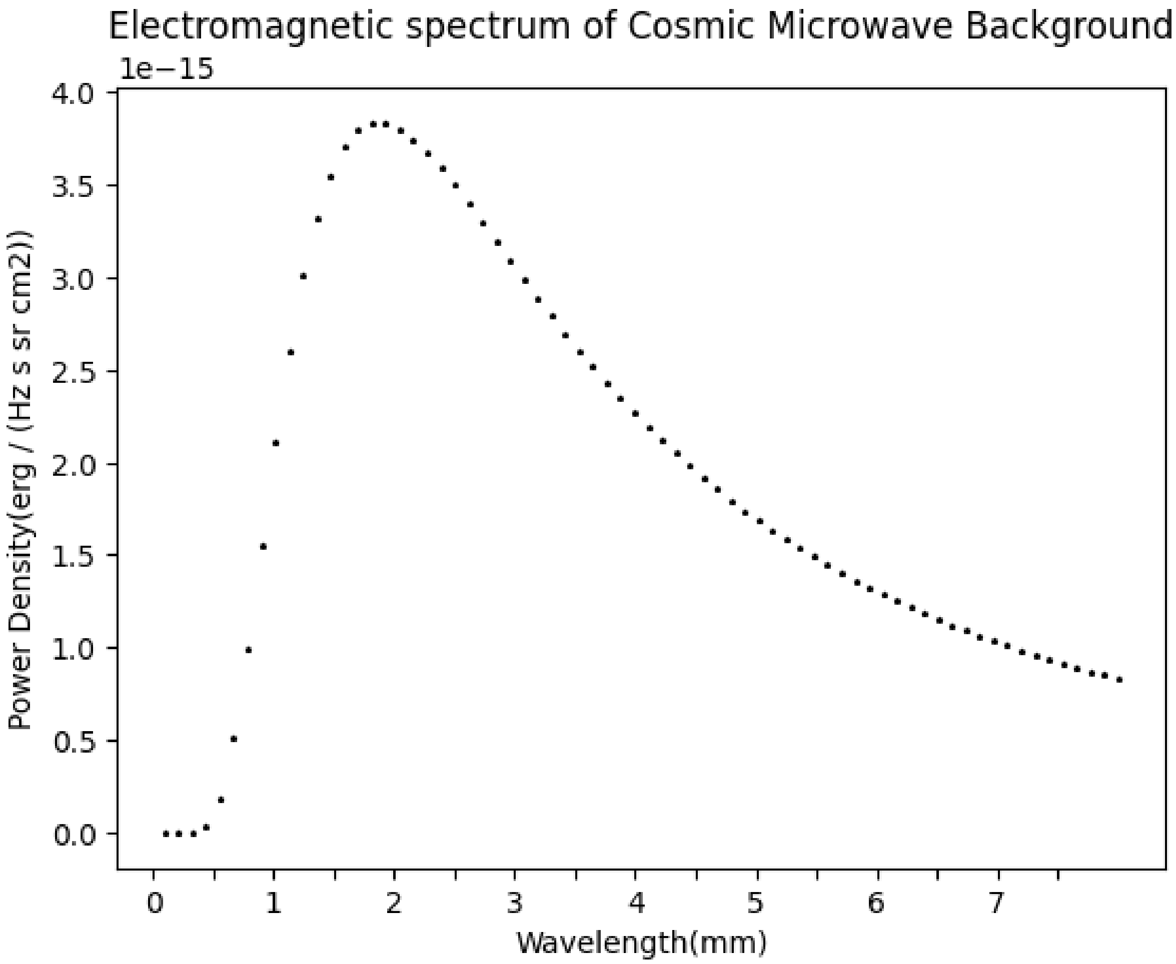
# 1 Back in my Days..

The cosmic microwave background radiation (CMB) is an electromagnetic signal measured in all directions, regardless of any light source present.

According to Big Bang Cosmology, the photons that make up the CMB are a relic of an early era in the universe, when it was in a hot, dense, opaque state, in constant equilibrium. The photons were released when the universe cooled to about 3000 Kelvin and was transparent to photons, known as the recombination era. These photons traveled untouched as the universe expanded and cooled before reaching us in its current state.

The spectrum of the CMB as measured now is shown below.




(a) Given this, how much has the universe expanded since the CMB photons were released? [2m]

**We are given the electromagnetic spectrum of the CMB now, which we are told used to follow the blackbody spectrum and remain untouched, thus, it can be assumed the current spectrum also follows a blackbody curve.**

**By measuring the graph we see that the peak wavelength of the current CMB to be around** $1.9mm = 1.9 \times 10^{-3}m$ **[0.5 mark]**

**By applying Wein's Displacement Law** $\lambda_{max} = \frac{b}{T}$ **we get the peak wavelength of the CMB during Recombination to be** $\frac{2.898\times10^{-3}mK}{3000K} = 9.66 \times 10^{-7}m$ **[0.5 mark]**

**Any change in wavelength is caused by the expansion of the universe, the ratio of the scale factor of the universe now against then is thus** $\frac{1.9\times10^{-3}m}{9.66\times10^{-7}m} = 1970$ **[1 mark]**

(b) During the time of recombination, the universe can be treated as a homogenous, isotropic fluid that is in thermal and energetic equilibrium. What was the energy density of radiation in the universe then? [5m]

Hint: Consider how all angles of light in a blackbody radiate through a chosen flat surface within a blackbody to fill its volume.

**By drawing out an arbitrary tiny flat surface within the blackbody universe, we know from the Stefan-Boltzman law that the radiation going through the surface follows the equation** $P = \sigma A T^4$ **[1 Mark], thus, the flux going through the surface is equal to** $j = \frac{P}{A} = \sigma T^4$

**In the case where photons are all going in one direction, the energy density of the blackbody u, follows the equation** $u = \frac{E}{V} = \frac{j\times A\times dt}{A\times dt\times v}$ **[1 Mark], thus** $j = u \times v$

**Considering radiation going in all angles which can be represented by a sphere, we consider** $\frac{2\pi R^2 sin\theta d\theta}{4\pi R^2}$ **[1 Mark] which is the proportion of photons in a specific ring around the sphere which all share a certain angle** $\theta$**, the photons travel through the small surface at different perpendicular speeds depending on their angle, following the formula** $v = c \times cos\theta$

**By integrating over all angles** $j = \int_0^{\frac{\pi}{2}} \frac{2\pi R^2 sin\theta d\theta}{4\pi R^2} u \times c \times cos\theta \, d\theta = \frac{c}{4} \times u$ **[1 Mark]**

**Thus, the energy density of a blackbody is** $u = \frac{4}{c}\sigma T^4$ **which in the case of the universe during recombination, gives** $u = \frac{4}{299792458m/s} 5.670 \times 10^{-8} W m^{-2} K^{-4} \times (3000K)^4 = 0.0613 J/m^3$ **[1 Mark]**

According to the ΛCDM model of the universe, the expansion of the universe is linked to its energy density, which is contributed by matter, radiation and dark energy, with dark energy having a constant energy density.

(c) Given that the current energy density of matter and dark energy is $2.82 \times 10^{-10} J/m^3$ and $6.00 \times 10^{-10} J/m^3$ respectively, what was the energy density of these 2 components during the era of recombination? [3m]

**According to the ΛCDM model, dark energy is linked to the cosmological constant, Λ which was first introduced as a constant by Einstein, thus it is treated as a constant with constant energy density [0.5 Mark], thus its density is always $\rho_\Lambda = 6.00 \times 10^{-10} J/m^3$. [1 Mark]**

**Matter density on the other hand follows the formula $\rho_{m,t} = \frac{E_0}{V_t} = \rho_{m,0} \times \frac{V_0}{V_t} = \rho_{m,0} \times a(t)^{-3}$, [0.5 Mark] thus when going back to the era of recombination, where $a(t_{rec}) = \frac{1}{1970}$, $\rho_{m,rec} = \rho_{m,0} \times (\frac{1}{1970})^{-3} = 2.82 \times 10^{-10} \times 1970^3 = 2.16 J/m^3$ [1 Mark]**

(d) Making the wild assumption that the universe's expansion is constant, what would have been the age of the universe during Recombination? [2m]

**The Hubble's constant which is currently equal to $H_0 = 67.80 kms^{-1}Mpc^{-1} = 2.20 \times 10^{-18} s^{-1}$ and follows the relationship $H = \frac{\frac{\delta a}{\delta t}}{a} = \frac{\dot{a}}{a}$. Given a universe with constant expansion rate, where $\dot{a} = k$ and $a = kt$, $H_0 = \frac{k}{a_0} = k = 2.20 \times 10^{-18} s^{-1}$ and the scale factor of the universe increases by $2.20 \times 10^{-18}$ per second. [1 Mark]**

**This gives us the classic estimation of the universe's age $\frac{1}{2.20 \times 10^{-18} s^{-1}} = 4.545 \times 10^{17} s = 14.4 Gyrs$ thus, during the era of recombination, when the scale factor is equal to $\frac{1}{1970}$ the universe must have been expanding for $\frac{\frac{1}{1970}}{2.20 \times 10^{-18} s^{-1}} = 2.307 \times 10^{14} s = 7.31 Myrs$ [1 Mark]**

## Brighter Than The Sun [Theory Q4] Solutions

(by Kane)

(a) (i) Note: amu not given, participants might use mass of a proton instead, or more accurately, $\frac{\text{g}}{\text{amu}} = N_A$

$$5.61 \times 10^{24}\ \text{kg/yr} = 1.7777 \times 10^{17}\ \text{kg/s}$$
$$= 1.7777 \times 10^{17}\ \text{kg/s} \times \frac{1\ \text{amu}}{1.6605 \times 10^{-27}\text{kg}}$$
$$= 1.0706 \times 10^{44}\ \text{amu/s}$$

$$\text{rate of helium nuclei created} = 1.0706 \times 10^{44}\ \text{amu/s} \times \frac{1\ \text{He atom}}{4.002603254\ \text{amu}}$$
$$= 2.6747 \times 10^{43}\ \text{He atoms/s}$$

$$\text{Energy per He atom} = \Delta mc^2$$
$$= (4 \times 1.007825031 - 4.002603254)\ \text{amu}\ c^2$$
$$= 0.02869687\ \text{amu}\ c^2$$
$$= 0.02869687\ \text{amu} \times \frac{1.6605 \times 10^{-27}\text{kg}}{1\ \text{amu}} c^2$$
$$= 4.7651 \times 10^{-29}\ \text{kg} \times (299792458\ \text{m/s})^2$$
$$= 4.2827 \times 10^{-12}\ \text{J}$$

$$L = 2.6747 \times 10^{43}\ \text{He atoms/s} \times \frac{4.2827 \times 10^{-12}\ \text{J}}{1\ \text{He atom}}$$
$$= 1.1455 \times 10^{32}\ \text{W}$$

(ii) Note: Fish marked this part

(1m) awarded for stating forms of efficiency loss [eg. via neutrinos], partial (0.5m) awarded for merely stating efficiency loss

(1m) awarded for stating stellar atmospheric absorption, fusion near core takes time to travel out etc also accepted

***Marker's comments:***
***Not well done.***
***Many did not clearly state what is meant by efficiency loss or merely stated one of the two reasons***

(b) Stefan Boltzmann: $L = 4\pi R^2 \sigma T^4$

$$T = \sqrt[4]{\frac{L}{4\pi R^2 \sigma}}$$
$$= \sqrt[4]{\frac{1.1455 \times 10^{32}\ \text{W}}{4\pi \times (12 \times 6.957 \times 10^8\ \text{m})^2 \times (5.67 \times 10^{-8}\ \text{W m}^{-2}\text{K}^{-4})}}$$
$$= 38972\ \text{K}$$

$$\lambda = \frac{b}{T} = \frac{2.898 \times 10^{-3}\ \text{m K}}{38972\ \text{K}} = 74.361\ \text{nm}$$

(c)

$$
\begin{aligned}
L_{\text{eddington}} &= 5.5 \times 10^4 \left( \frac{M}{M_\odot} \right) L_\odot \\
\frac{M}{M_\odot} &= \frac{L_{\text{eddington}}}{5.5 \times 10^4 \times L_\odot} \\
&= \frac{1.1455 \times 10^{32} \text{ W}}{5.5 \times 10^4 \times 3.828 \times 10^{26} \text{ W}} \\
&= 5.4408 \\
M &= 5.4408 M_\odot \\
&= 1.0822 \times 10^{31} \text{ kg}
\end{aligned}
$$

WR mass = $M_{\text{WR}} = 5.4408 M_\odot$

WD mass = $M_{\text{WD}} = 1.28 M_\odot$

Let's say WD, WR mass doesn't change significantly during accretion

$q = \frac{M_{\text{WD}}}{M_{\text{WR}}} \frac{5.4408}{1.28} = 4.250625$

$$
\begin{aligned}
M_{\text{enclosed inside WR}}(r) &= \int_0^r \rho(r) \pi r^2 \text{ d}r \\
&= \int_0^r \frac{k}{r^2} \pi r^2 \text{ d}r \\
&= \int_0^r k\pi \text{ d}r \\
&= k\pi r \propto r \\
\frac{M_{\text{enc}(r)}}{M_{\text{total}}} &= \frac{r}{R}
\end{aligned}
$$

Let's make the simplifying assumption that the WD slowly slurps all the star's atmosphere outside it's roche lobe (approximated as a hill sphere), then the star's atmosphere magically expands back afterwards.

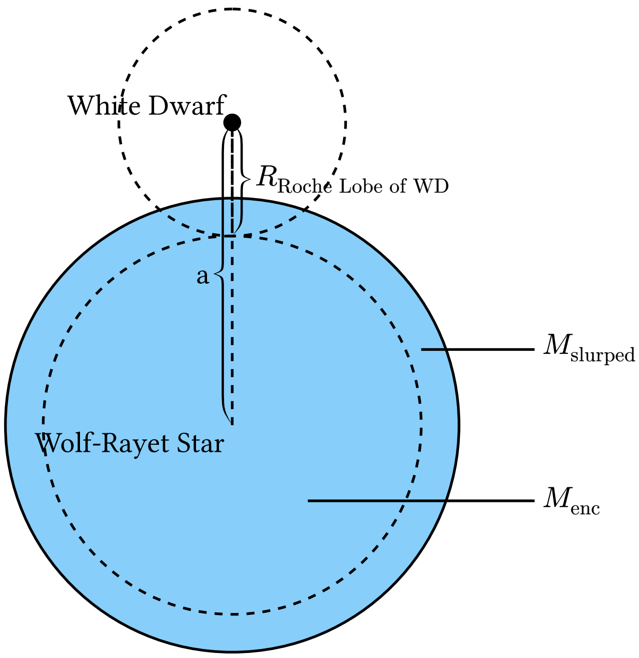



Figure 1: Why did I draw a whole ass diagram (not to scale)

$$
\begin{aligned}
M_{\text{slurped}} &= M_{\text{WR}} - M_{\text{enc}} \\
\frac{M_{\text{slurped}}}{M_{\text{WR}}} &= 1 - \frac{M_{\text{enc}}}{M_{\text{WR}}} \\
\frac{10^{-4} M_{\odot} \times 64}{5.4408 M_{\odot}} &= 1 - \frac{a - R_{\text{Roche Lobe of WD}}}{R_{\text{WR}}} \\
\frac{a - R_{\text{Roche Lobe of WD}}}{R_{\text{WR}}} &= 1 - \frac{10^{-4} M_{\odot} \times 64}{5.4408 M_{\odot}} \\
&= 0.99883 \\
a - R_{\text{Roche Lobe of WD}} &= 0.99883 R_{\text{WR}} \\
a - a \frac{0.49 q^{\frac{2}{3}}}{0.6 q^{\frac{2}{3}} + \ln\left(1 + q^{\frac{1}{3}}\right)} &= 0.99883 R_{\text{WR}} \\
a - 0.5067 a &= 0.99883 R_{\text{WR}} \\
a &= \frac{0.99883}{0.4933} R_{\text{WR}} \\
&= 2.0247 \times (12 R_{\odot}) = 24.072 R_{\odot} \\
&= 2.0247 \times (12 \times 6.957 \times 10^{8} \text{ m}) \\
&= 1.6904 \times 10^{10} \text{ m}
\end{aligned}
$$

# Medium 1

## Question

JWST launched on December 25 2021 from Kourou, French Guiana 5°N.

**a.** Assuming that earth is a perfect sphere of radius $6.370 \times 10^6$ m, find the tangential velocity at the launch site due earth's rotation. [2m]

To plan out the trajectory of JWST to its final position around Lagrange Point 2, Scientist A wanted to make a rough calculation using Satellite Bob, a simulated model.

Satellite Bob is in orbit around the earth.

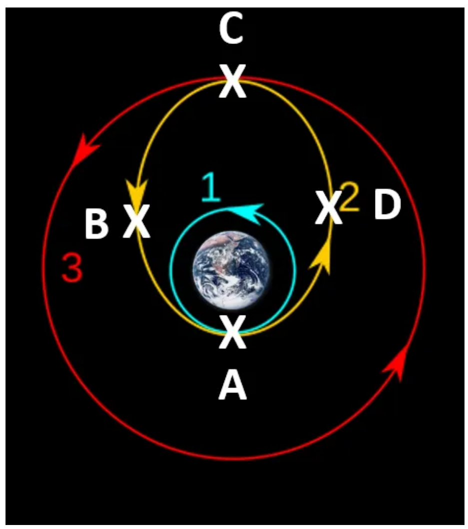



Figure 1: Satellite Bob's transfer orbit

*https : //www1.grc.nasa.gov/beginners − guide − to − aeronautics/earth/*
*https : //commons.wikimedia.org/wiki/File :* $Hohmann_transfer_orbit.svg$

**b.** There are two burns conducted in order for a Hohmann transfer for Satellite Bob, which starts at the innermost orbit, orbit 1. Write in order, the burns conducted. eg. E, F (as marked by the crosses and letters A–D in the image). [2m]

**c.** Satellite Bob's orbit 1 is 2000 km from the surface of the earth and orbit 2 is 40 000 km from the surface of earth, find the $\Delta v$ made during the first and second Hohmann transfers. List your answer in the format ($\Delta v$ of transfer 1, $\Delta v$ of transfer 2). (Both orbits are circular). [7m]

**d.** Calculate the total energy of Satellite Bob at Orbit 3, given the Mass(satellite) is 6500 kg. [3m]

**e.** JWST is set to orbit around Lagrange point 2 (L2). Find an equation describing the balance of forces at L2. Express your ans in terms of $G$, $M_\odot$, $M_{Earth}$, $R$, and $x$. [8m]

One of JWST's tasks is to observe exoplanets, to learn more about their atmospheric composition. One example is the WASP-39 system. Here, we'll discuss the study of WASP-39b, one of the planets in this system. The following is a transmission spectrum of the transit of WASP-39b. (only for visual reference, don't use numbers from the image to answer the following questions)

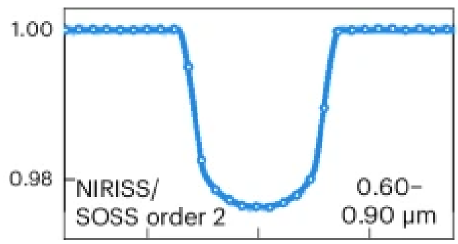



Figure 2: Transmission spectrum of WASP-39b transit - relative flux against time. Only for visual reference, don't use numerical values in the image to answer.
https://www.nature.com/articles/s41550-024-02292-x

The following parameters will be useful for solving parts f and g.

| | |
|---|---|
| Mass of WASP-39 | 0.93 $M_\odot$ |
| Mass of WASP-39b | 0.28 $M_{Jupiter}$ |
| Radius of WASP-39b | 1.27 $R_{Jupiter}$ |
| Seperation of WASP-39 from WASP-39b as viewed from earth (edge-on view) | 0.047$AU$ |
| Peak wavelength of WASP-39 | 537$nm$ |
| Orbital inclination of WASP-39b (as viewed by jwst) | 87.83° |
| Distance from earth to WASP-39 | 700$ly$ |
| Ratio of transit distance to diameter of WASP-39 | 0.92388 |

**f.** Assume the transit lasted 120 min from first entry of the Wasp39b to the complete exit.Find the radius of the star WASP-39. [3m]

**g.** Find the flux received by WASP-39b from WASP-39 [3m]

## Answers

- (a)

  $V = \frac{2\pi 6.371\times10^6}{86164.1}\times cos5$ (1m)
  V=463 m $s^{-1}$ (1m)
  Well done. Easy question to solve.

- (b) A, C (1m for each ans)
  Also well done, again an easy question.

- (c) $(-2.08\times10^3 ms^{-1}, 1.31\times10^3 ms^{-1})$ (1m)

  $\Delta v_1 = \sqrt{GM\times(\frac{2}{2000\times10^3+6.371\times10^6}-\frac{2}{40000\times10^3+2000\times10^3+6.371\times10^6\times2}}(2m)-\sqrt{\frac{GM}{2000\times10^3+6.371\times10^6}}(1m)$

  $\Delta v_2 = \sqrt{\frac{GM}{40000\times10^3+6.371\times10^6}}-\sqrt{GM\times(\frac{2}{40000\times10^3+6.371\times10^6}-\frac{2}{40000\times10^3+2000\times10^3+6.371\times10^6\times2}}$

  (3m for each for eqns, 1m for correctl ans)

  Many knew the vis viva eqn and were able to get the correct formulae. Hence partial credit was awarded. However, many participants did not take into account the radius of the earth in their calculations and hence got the wrong answer.

- (d) Total energy $= -2.80\times10^{10}$ J (1m for kinetic term, 1m for potential term, 1m for ans) $E_{potential} = -\frac{GMm}{r}$ $E_{kinetic} = \frac{GMm}{2r}$ $E_{total} = -\frac{GMm}{2r}$
  Also well done. Ecf was awarded (3m, if alternative ans is correct) from part c if participant did not include the radius of the earth.

- (e)

  $$\frac{GM_S}{(r+x)^2}+\frac{GM_E}{x^2}=\frac{G(M_S+M_E)}{r^3}(r+x)$$

  (1m for each term - left and right - and 1m for correct form including sign)
  Many participants were able to show the equations for gravitational forces from the sun and the earth but were unable to get the centripetal force component in the correct form. Hence, most participants were awarded 2m.

- (f)$8.94\times10^9 m$

  Time taken to traverse star
  $\frac{120\times60}{sin87.3}$−time taken for planet to move its diameter newline
  $T^2 = \frac{4\pi a^3}{GM}$
  T = 334 840s (1m)

Time taken for planet to move its diameter
$\frac{d}{2\pi a} \times T = 1342.3$ (0.5m for working)

Distance traveled by planet = time taken to traverse star/$T \times 2\pi a$
$= 1.651 \times 10^{10}$ (0.5m)

$\frac{s}{2r_{star}} = 0.92388$ (0.5m)
$r_{star} = 8.9353 \times 10^9 m$
$= 8.94 \times 10^9 m$ (0.5m)

marks are also awarded if candidate does $\frac{1}{2}(\frac{0.047AU}{0.93288}) = 3.81 \times 10^9 m$ 1m for ans, 2m for method (1m method if 1/2 is not included)

Both parts f and g were interpreted differently compared to what I intended. Most participants understood it as the distance travelled by WASP-39b during the transit. Hence, almost all participants were not marked based on the first answer.

I accepted the different ways participants interpreted these two questions and marked them by accuracy of the steps taken and the answer.

Many participants did well for this question.

- (g) $2.7 \times 10^4 W m^{-2}$

$L = 4\pi r_{star}^2 e\sigma T^4$
$T = b/\lambda_{peak}$ (1m)
$I = \frac{L}{4\pi a^2}$ (1m)
$= 2.7 \times 10^4 W m^{-2} (1m)$
2m marks are awarded if candidate multiplies by $\pi r_{WASAP39-b}^2$

Due to the alternative interpretations in part f, some participants went to find the length of the semimajor axis of WASP-39b's orbit.

I gave method marks for the temperature and flux equations and gave answer marks if the values substituted were accurate, esp the semimajor axis.

Here comes the second misinterpretation. Many participants ended up finding the flux/intensity multiplied by the perpendicular surface area of WASP-39b exposed to light from WASP-39, wrongly thinking that the result of the product is flux. Hence they calculated power, not intensity as required. However, method marks (2m) were still awarded as this is an additional step taken from finding the intensity/flux received by WASP-39b.