2026 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 15 eligible scorers on the NAC will be invited to the training camp. A third round will be conducted during the camp to select the national team for the IOAA 2026. In order to qualify for camp, you must be a high school student with US citizenship or permanent residency.

The test must be completed within 2.5 hours (150 minutes) and the maximum number of points is **300 points**. The point breakdown per problem is as follows.

Problem	P1	P2	P3	P4	P5	P6	P7	P8	P9	**Total**
Points	10	15	15	20	20	40	40	55	85	**300**

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. Some problems may require a designated answer sheet; these will be provided. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific calculator (non-programmable and non-graphing) for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on April 20, 2026.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Students must be proctored during the entire duration of the exam.

This exam sheet and your answer sheets (including scratch papers) should be returned to your proctors once the exam ended. Your proctors should upload your answer sheets to the Google Form provided to them.

We acknowledge the following people for their contributions to this year's exam:

Srihari (Hari) Balaji, Feodor Yevtushenko, Ferdinand, Hagan Hensley, Joe McCarty, Lucas Pinheiro, Vincent Bian, Elizabeth Lee

After reading the instructions, please make sure to sign, affirming that:

1. All work on this exam has been completed by me.
2. I took this exam under the supervision of a proctor.
3. I did not receive any external help beyond the materials provided.
4. I will not discuss the contents of this exam with anyone until April 20, 2026.
5. Failure to follow these rules will result in disqualification from the exam.

Student Signature: ______________________ Date: ______________________

1. **(10 points)** *A "gravitational atom"*

   Bohr's semiclassical model of the hydrogen atom assumed that the ground state angular momentum of the electron is $\hbar = h/(2\pi)$. While this model is incorrect, the Bohr radius does give the correct lengthscale of the hydrogen atom. By analogy to the Bohr model, consider a bound system consisting of a neutron and an electron interacting purely gravitationally. If the orbital angular momentum is $\hbar$, what would be the radius of this "atom"?

   Solution:

   This question was inspired by a comment in Professor Ethan Neil's quantum mechanics class.

   The electron mass is sufficiently small compared to that of the neutron that we can neglect the fact that both particles orbit about their combined center of mass and treat the neutron as stationary, although it is not difficult to take that into account using the reduced mass. Setting the gravitational force equal to centripetal acceleration with $L = m_e v r = \hbar$ **(2 points)**,

   $$\frac{G m_n m_e}{r^2} = \frac{m_e v^2}{r} = \frac{\hbar^2}{m_e r^3}, \tag{1}$$

   (2 points) so

   $$\begin{aligned} r = \frac{\hbar^2}{G m_n m_e^2} &= \frac{(6.626 \times 10^{-34}\,\mathrm{J\,s})^2}{4\pi^2 (6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})(1.675 \times 10^{-27}\,\mathrm{kg})(9.110 \times 10^{-31}\,\mathrm{kg})^2} \\ &= \boxed{1.2 \times 10^{29}\,\mathrm{m}} \cdot \frac{1\,\text{light year}}{9.461 \times 10^{15}\,\mathrm{m}} = \boxed{1.3 \times 10^{13}\,\text{light years}}, \end{aligned} \tag{2}$$

   which is significantly larger than the observable universe! This illustrates the relative weakness of the gravitational force compared to electromagnetism. **(3 points for final expression for r, 3 points for final calculation - meters is fine)**

2. **(15 points)** *Celestial Sphere: Night at the Observatory*

   An observer is located at latitude $\phi = +42.0^\circ$ (Northern Hemisphere). Assume the Earth is a perfect sphere and ignore atmospheric refraction.

   Star A has equatorial coordinates

   $$\alpha = 6^{\mathrm{h}}40^{\mathrm{m}}, \qquad \delta = +62.0^\circ.$$

   On a particular night, the local sidereal time (LST) is

   $$\mathrm{LST} = 5^{\mathrm{h}}10^{\mathrm{m}}$$

   at 10:00 PM local clock time.

   You may assume that one sidereal day equals $23^{\mathrm{h}}56^{\mathrm{m}}$ of clock time.

   (a) **(2 points)** Compute the hour angle H of star A at 10:00 PM. State whether the star is east or west of the meridian at that moment.

   (b) **(5 points)** Compute the altitude h of star A at 10:00 PM. Give your answer in degrees.

   (c) **(3 points)** How much clock time will pass after 10:00 PM until star A reaches upper culmination? Express your answer in hours, minutes, and seconds.

   Suppose star B has declination

   $$\delta = +30.0^\circ.$$

(d) **(5 points)** For star B, compute the total clock-time duration during which the star is above the horizon during one sidereal day. Express your final answer in hours and minutes of clock time.

Solution:

Given $\phi = +42.0^\circ$, $\delta = +62.0^\circ$, and $\alpha = 6^h40^m$. At 10:00 PM, LST $= 5^h10^m$. Recall $H = \text{LST} - \alpha$ and $H > 0$ means west of the meridian.

(a) **Hour angle and sky side. (2 points)**

Convert to hours:

$$\alpha = 6^h40^m = 6.666\overline{6}\ \text{h}, \qquad \text{LST} = 5^h10^m = 5.166\overline{6}\ \text{h}.$$

Thus

$$H = 5^h10^m - 6^h40^m = -1^h30^m.$$

Since $H < 0$, the star is *east* of the meridian.

Equivalently, in degrees (15° per hour):

$$H = -1.5\ \text{h} \times 15^\circ/\text{h} = -22.5^\circ.$$

Grading:

- 2 pts: Correct H and correct east/west interpretation.
- 1 pt: Correct subtraction but wrong east/west interpretation.
- 1 pt: Arithmetic error in conversion but correct method.
- 0.5 pt: Correct formula $H = \text{LST} - \alpha$ but major arithmetic mistake.
- 0 pts: Incorrect formula or no meaningful progress.

Accept answers in range: $-22.5^\circ \pm 0.3^\circ$.

(b) **Altitude at 10:00 PM. (5 points)**

Using spherical law of cosines,

$$\sin h = \sin\phi \sin\delta + \cos\phi\cos\delta\cos H. \tag{3}$$

Substitute $\phi = 42^\circ$, $\delta = 62^\circ$, $H = -22.5^\circ$:

$$\sin h = \sin 42^\circ \sin 62^\circ + \cos 42^\circ \cos 62^\circ \cos(22.5^\circ).$$

Numerically,

$$\sin h \approx 0.9131$$

$$h \approx \arcsin(0.9131) \approx 65.9^\circ.$$

$$\boxed{h \approx 65.9^\circ}$$

Accept answers between 65° and 67°.

Grading:

- 5 pts: Correct substitution, trig evaluation, and final altitude.
- 4 pts: Correct setup with minor arithmetic error.
- 3 pts: Correct formula but computational errors.
- 2 pts: Some progress towards Eq. 3 but incorrect substitution and arithmetic errors.
- 1 pt: Partial substitution or wrong angle sign.
- 0 pts: No correct use of relation.

Note: If a student forgets $\cos(-H) = \cos H$, do not deduct heavily unless the conceptual reasoning is affected.

(c) **Clock time to upper culmination. (3 points)**

Upper culmination occurs when $H = 0$, i.e. when LST = α.

Currently $H = -1^{\mathrm{h}}30^{\mathrm{m}}$, so the LST must increase by

$$\Delta H = 1^{\mathrm{h}}30^{\mathrm{m}} = 1.5 \text{ sidereal hours.}$$

A sidereal day is 24 sidereal hours = $23^{\mathrm{h}}56^{\mathrm{m}}$ of clock time.

Thus

$$1 \text{ sidereal hour} = \frac{23^{\mathrm{h}}56^{\mathrm{m}}}{24} \approx 59^{\mathrm{m}}50^{\mathrm{s}}.$$

Therefore

$$\Delta t = 1.5 \times 59^{\mathrm{m}}50^{\mathrm{s}} \approx 1^{\mathrm{h}}29^{\mathrm{m}}45^{\mathrm{s}}.$$

$$\boxed{1^{\mathrm{h}}29^{\mathrm{m}}45^{\mathrm{s}}}$$

Grading:

- 3 pts: Correct sidereal interval and correct elapsed time.
- 2 pts: Correct sidereal interval and recognizes conversion needed but performs it incorrectly.
- 1 pt: Correct sidereal interval but assumes 1 sidereal hour = 1 clock hour, leading to $1^{\mathrm{h}}30^{\mathrm{m}}$.

(d) **Time above the horizon. (5 points)**

Use the horizon condition

$$\cos H_0 = -\tan\phi \tan\delta,$$

which comes from Eq. 3 with altitude $h = 0$.

Substitute $\phi = 42^\circ$, $\delta = 30^\circ$:

$$\cos H_0 = -\tan 42^\circ \tan 30^\circ.$$

Using

$$\tan 42^\circ \approx 0.900, \qquad \tan 30^\circ \approx 0.577,$$

$$\cos H_0 \approx -0.519.$$

Thus

$$H_0 \approx \cos^{-1}(-0.519) \approx 121.3^\circ.$$

Convert to hours:

$$H_0 = \frac{121.3^\circ}{15^\circ/\text{hour}} \approx 8.09 \text{ hours}.$$

The star is above the horizon for

$$2H_0 \approx 16.18 \text{ sidereal hours}.$$

Convert to clock time using

$$1 \text{ sidereal hour} \approx 59^\text{m}50^\text{s}.$$

$$16.18 \times 59^\text{m}50^\text{s} \approx 16^\text{h}08^\text{m}.$$

$$\boxed{\text{Star is above the horizon for about } 16^\text{h}08^\text{m}.}$$

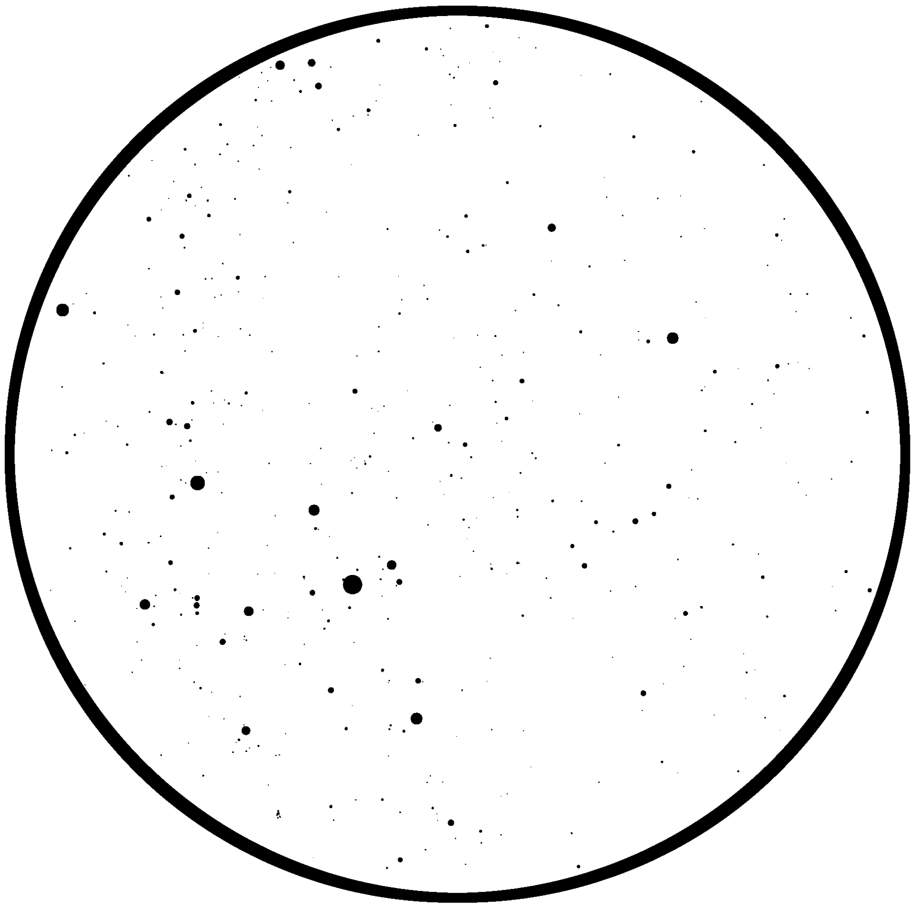
Grading:

- 5 pts: Correct setup of the horizon equation, correct computation of H_0, and correct duration above the horizon.
- 4 pts: Correct method but minor arithmetic error in H_0 or time conversion.
- 3 pts: Correct horizon equation but significant computational errors.
- 2 pts: Partial substitution into the horizon condition.
- 1 pt: Incorrect derivation of horizon condition from Eq. 3.
- 0 pts: No correct use of the horizon relation.

3. **(15 points)** *Sky Map Attack!*

Please answer this question in the provided Question 3 Answer Sheet

Before IOAA 2026, you and your friends decided to go on vacation to a deserted island. One night, you looked up at the sky and it looked like the image below:



As the sky was bright, you and your friends gave each other quizzes about the sky.

(a) **(3 points)** What is the estimated latitude of the island?
Hint: The all sky map above is a stereographic projection, which means the zenith distance would follow: $z = 2\arctan(r/R)$*, where* r *is the distance of an object to the center of the map and* R *is the radius of the all sky map.*

(b) **(2 points)** Mark and label the cardinal directions: North, East, South, and West.

(c) **(2 points)** Draw and label the celestial equator.

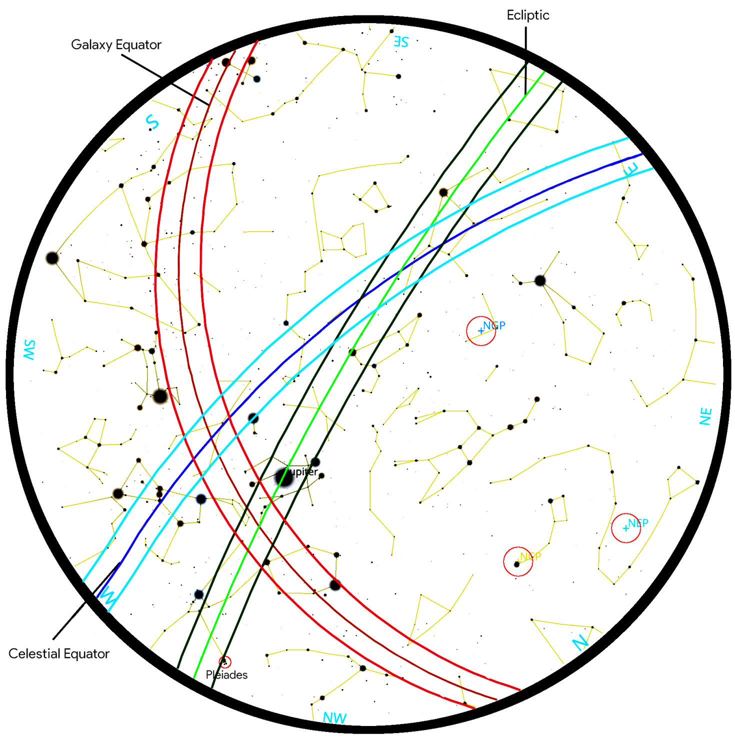
(d) **(2 points)** Draw and label the ecliptic.

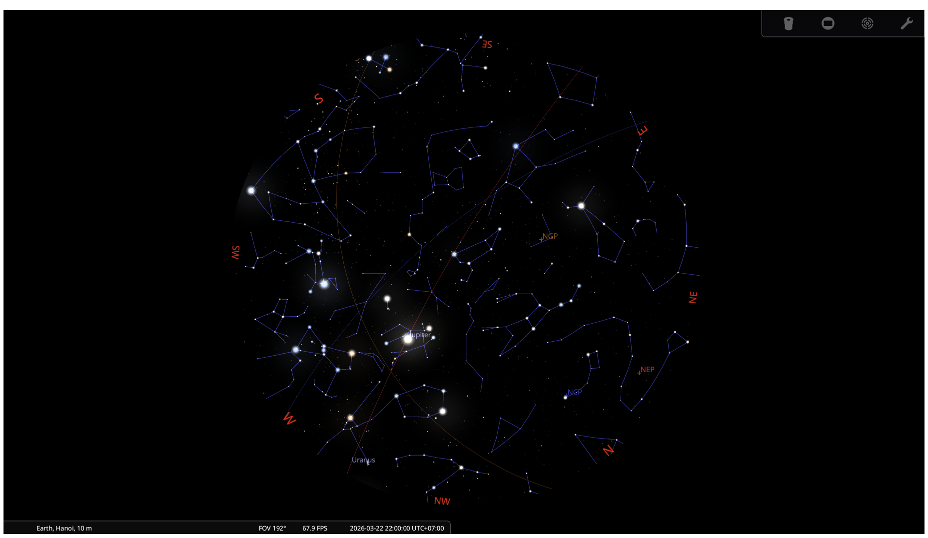
(e) **(3 points)** Mark and label (North/South) the available Celestial, Ecliptic, and Galactic Poles.

(f) **(1.5 points)** Are there any planets seen in the sky? If yes, mark and label the planets.

(g) **(1.5 points)** Circle and label the Pleiades (M45).

Solution: Please refer to these two images below for the sky map. The first image is the inverted image, while the second image is the screenshot from Stellarium.





(a) In the sky image, we can see that there is the big dipper and polaris, which means it is in the northern hemisphere. The Polaris altitude is around $a_{\text{Polaris}} = 2.5$ cm and the distance to the center of the sky is around $R = 8$ cm. So, using the zenith distance formula from the hint, we get the latitude $\approx \varphi = 90^\circ - 2\arctan(r/R) = 90^\circ - 2\arctan(5.5/8) \approx 21^\circ$N. In this subpart, a $18^\circ - 24^\circ$N gets the full 3 points; a $15^\circ - 17^\circ$N or $25^\circ - 27^\circ$N gets 1.5 points; and outside that range gets no points. If you didn't write your thoughts, you'd only get 1 point if it's within $18^\circ - 24^\circ$N.

Note: As you can see in the Stellarium image, the sky map was actually taken in Hanoi, Vietnam. Hence, the true latitude is around $21°$N.

(b) Marked and labeled in the sky map images. *Each cardinal point is 0.5 point.*

(c) Drawn and labeled in the sky map images. As long as it's within the range drawn out in the map, get full 2 points.

(d) Drawn and labeled in the sky map images. As long as it's within the range drawn out in the map, get full 2 points.

(e) Marked and labeled in the sky map images. As long as the positions are within the red circles, get full points; each pole is one point.

(f) There is only one planet: Jupiter. It's also marked and labeled in the sky map images. 1 points for marked correctly, 0.5 points for label Jupiter correctly. 0 points if marked/labeled more than 1 planet.t

(g) Circled and labeled in the inverted sky map. Either full 1.5 points if marked correctly or zero.

4. **(20 points)** *Solar Observations*

An astronomer who enjoys solar observations has the habit of recording the Sun's right ascension and declination every day. These coordinates are always recorded at the same civil time (the time on the clock). Between two consecutive observations, which happened when the Sun was in the Northern Celestial Hemisphere, the declination of the Sun increased by $0.3650°$. Estimate the days in which these two observations occurred.

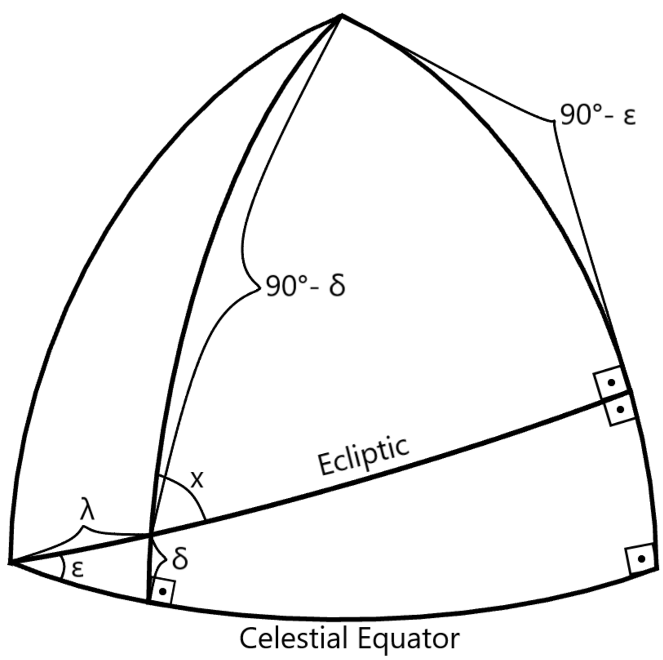
Neglect the eccentricity of Earth's orbit.

Solution:

First Approach:

In order to estimate the days of the observations, the first step is to determine the declination of the Sun at the point in which its rate of variation in declination is $0.3650°$/day. Note that there are two such points in the celestial sphere, but since the problem statement specifies that the Sun was in the Northern Celestial Hemisphere, it is possible to obtain an unique solution.

Using the figure below, it is possible to obtain an expression for angle x, which can be used to break down the angular velocity of the Sun into a right ascension component and a declination component.



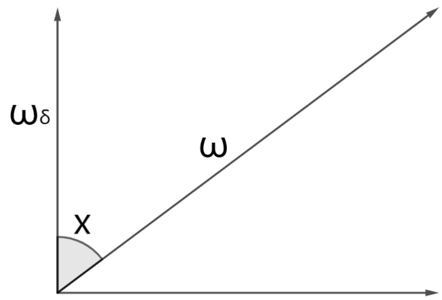
Using the spherical law of sines:

$$\frac{\sin x}{\sin(90° - \varepsilon)} = \frac{\sin 90°}{\sin(90° - \delta)}$$
$$\sin x = \frac{\cos \varepsilon}{\cos \delta}$$

Using the Pythagorean trigonometric identity:

$$\cos x = \sqrt{1 - \left(\frac{\cos \varepsilon}{\cos \delta}\right)^2}$$

Neglecting the eccentricity of the Earth's orbit, the angular velocity of the Sun on the Ecliptic corresponds to $\omega = \frac{360^\circ}{365.2564 \text{ days}} = 0.98561^\circ/\text{day}$.



The declination component of the angular velocity (ω_δ) can be obtained through the following expression:

$$\begin{aligned}
\omega_\delta &= \omega \cos x \\
\frac{\omega_\delta}{\omega} &= \sqrt{1 - \left(\frac{\cos \varepsilon}{\cos \delta}\right)^2} \\
\frac{\cos \varepsilon}{\cos \delta} &= \pm\sqrt{1 - \left(\frac{\omega_\delta}{\omega}\right)^2} \\
\cos \delta &= \pm \frac{\cos \varepsilon}{\sqrt{1 - \left(\frac{\omega_\delta}{\omega}\right)^2}} \\
\cos \delta &= \pm \frac{\cos 23.44^\circ}{\sqrt{1 - \left(\frac{0.365}{0.98561}\right)^2}} \\
\cos \delta &= \pm 0.9877
\end{aligned}$$

Since the Sun is in the Northern Celestial Hemisphere and the change in declination is positive, the Sun must be in the first quadrant of the Ecliptic. Therefore:

$$\begin{aligned}
\cos \delta &= 0.9877 \\
\delta &= 8.995^\circ
\end{aligned}$$

The next step is to find the ecliptic longitude of the Sun. Using the spherical law of sines:

$$\begin{aligned}
\frac{\sin \lambda}{\sin 90^\circ} &= \frac{\sin \delta}{\sin \varepsilon} \\
\sin \lambda &= \frac{\sin 8.995^\circ}{\sin 23.44^\circ} \\
\sin \lambda &= 0.3930 \\
\lambda &= 23.14^\circ
\end{aligned}$$

Since the angular velocity of the Sun throughout the ecliptic is constant if Earth's orbital eccentricity is constant, it is possible to obtain the number of days since the vernal equinox:

$$\begin{aligned}\lambda &= \omega d\\ d &= \frac{\lambda}{\omega}\\ d &= \frac{23.14^\circ}{0.98561^\circ/\text{day}}\\ d &= 23.48 \text{ days}\end{aligned}$$

Therefore, the observation corresponds to days 23 and 24 after the vernal equinox. Since the vernal equinox occurs around March 20$^{\text{th}}$, the observations correspond to April 12$^{\text{th}}$ and April 13$^{\text{th}}$.

Since the equinox does not necessarily happen on March 20$^{\text{th}}$, pairs of days between April 10$^{\text{th}}$ and April 15$^{\text{th}}$ are acceptable without any point deductions.

Second Approach:

It is also possible to solve this problem in an alternative way. Consider the following expression, which can be obtained through the law of sines for spherical triangles:

$$\sin\delta = \sin\varepsilon \sin\lambda$$

Deriving this expression with respect to time:

$$\begin{aligned}\cos\delta \frac{\mathrm{d}\delta}{\mathrm{d}t} &= \sin\varepsilon\cos\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t}\\ \sqrt{1-\sin^2\delta}\frac{\mathrm{d}\delta}{\mathrm{d}t} &= \sin\varepsilon\cos\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t}\\ \sqrt{1-(\sin\varepsilon\sin\lambda)^2}\frac{\mathrm{d}\delta}{\mathrm{d}t} &= \sin\varepsilon\cos\lambda\frac{\mathrm{d}\lambda}{\mathrm{d}t}\\ [1-(\sin\varepsilon\sin\lambda)^2]\left(\frac{\mathrm{d}\delta}{\mathrm{d}t}\right)^2 &= \sin^2\varepsilon\cos^2\lambda\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2\end{aligned}$$

The rate $\frac{\mathrm{d}\lambda}{\mathrm{d}t}$ is constant and corresponds to $\frac{360^\circ}{365.2564 \text{ days}} = 0.98561^\circ/\text{day}$. Based on the problem statement, $\frac{\mathrm{d}\delta}{\mathrm{d}t} = 0.3650^\circ/\text{day}$. In order to simplify the expression, it is convenient to define a variable $k = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 \Big/ \left(\frac{\mathrm{d}\delta}{\mathrm{d}t}\right)^2 \approx 7.29$.

Therefore:

$$
\begin{aligned}
1 - \sin^2 \varepsilon \sin^2 \lambda &= k \sin^2 \varepsilon \cos^2 \lambda \\
1 &= \sin^2 \varepsilon (\sin^2 \lambda + k \cos^2 \lambda) \\
\frac{1}{\sin^2 \varepsilon} &= \sin^2 \lambda + k(1 - \sin^2 \lambda) \\
\frac{1}{\sin^2 \varepsilon} &= \sin^2 \lambda + k - k \sin^2 \lambda \\
\frac{1}{\sin^2 \varepsilon} &= (1-k) \sin^2 \lambda + k \\
\sin^2 \lambda &= \frac{1}{1-k} \left(\frac{1}{\sin^2 \varepsilon} - k \right)
\end{aligned}
$$

Since the declination increased between the two observations and the Sun was in the Northern Celestial Hemisphere, λ must be in the first quadrant. Therefore:

$$
\begin{aligned}
\sin^2 \lambda &= 0.1545 \\
\sin \lambda &= 0.3930 \\
\lambda &= 23.14^\circ
\end{aligned}
$$

From this point onward, the solution follows the same steps as the first approach.

- **Rubrics - First Approach:**
  - **(7 points)** correctly drawing or describing the geometry of the problem.
  - **(4 points)** finding an expression for the cosine of x.
  - **(1 point)** correctly breaking down the angular velocity of the Sun to find the declination component.
  - **(2 points)** determining the declination of the Sun.
  - **(2 points)** determining the ecliptic longitude of the Sun.
  - **(2 points)** determining the number of days since the vernal equinox.
  - **(1 point)** estimating a reasonable day for the vernal equinox.
  - **(1 point)** correct final answer.
- **Rubrics - Second Approach:**
  - **(1 point)** writing down the expression that correlates $\sin \delta$, $\sin \varepsilon$, and $\sin \lambda$.
  - **(3 points)** correctly deriving the expression with respect to the time.
  - **(12 points)** correctly simplifying the expression to obtain the value of λ.
  - **(2 points)** determining the number of days since the vernal equinox.
  - **(1 point)** estimating a reasonable day for the vernal equinox.
  - **(1 point)** correct final answer.

5. **(20 points)** *Enceladus geysers*

   Enceladus is a moon of Saturn which has an outer crust of ice and is believed to have a subsurface liquid water ocean. Here are some useful physical properties:

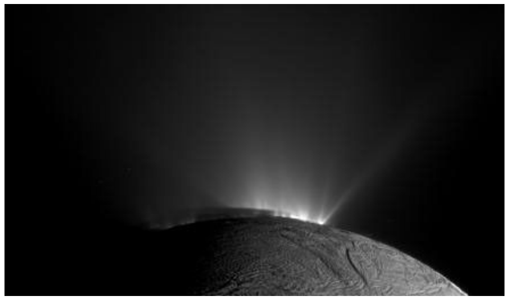


Figure 1: Cassini image of geysers from Enceladus. From NASA.

- Radius of Enceladus (including the ice shell) $R = 250\,\mathrm{km}$
- Ice shell thickness $H = 25\,\mathrm{km}$
- Density of ice $\rho_i = 920\,\mathrm{kg/m^3}$
- Surface gravity $g = 0.11\,\mathrm{m/s^2}$ (you can assume this is constant throughout the icy shell to the top of the subsurface ocean)
- Semi-major axis of Enceladus' orbit around Saturn $a = 2.4 \times 10^8\,\mathrm{m}$
- Mass of Saturn $M = 5.7 \times 10^{26}\,\mathrm{kg}$

(a) **(10 points)** The Cassini spacecraft observed that geysers of water erupt from the surface at a speed of $v = 400\,\mathrm{m/s}$ relative to Enceladus. Calculate whether this water has sufficient speed to escape the gravitational field of Enceladus. If so, will it remain gravitationally bound to Saturn or escape into solar orbit?

(b) **(6 points)** Assuming that the water in the geysers starts from rest at the top of the subsurface ocean, calculate the pressure necessary to accelerate the water to $v = 400\,\mathrm{m/s}$ at the surface of Enceladus, where there is no ambient pressure. Assume that viscous losses are negligible and the geyser material has a constant density of $1000\,\mathrm{kg/m^3}$, ignoring any phase changes.

(c) **(4 points)** Estimate the hydrostatic pressure due to the weight of the icy shell at the top of the subsurface ocean. Is that pressure sufficient to explain the speed of the geysers?

Solution:

This question was inspired by a homework problem in Professor Brad Hindman's fluid dynamics class.

(a) We are not given the mass of Enceladus, but we know that $g = Gm/R^2$. The escape velocity of Enceladus is **(1 point for escape velocity formula, 2 points for rewriting in terms of known quantities, 2 points for calculation)**

$$v_{esc,E} = \sqrt{\frac{2Gm}{R}} = \sqrt{2gR} = \sqrt{2(0.11\,\mathrm{m/s^2})(250 \times 10^3\,\mathrm{m})} = 230\,\mathrm{m/s} < v, \tag{4}$$

so $\boxed{\text{the geyser can escape Enceladus}}$ **(1 point)**.

The escape velocity of Saturn from Enceladus is **(1 point for formula, 1 point for calculation)**

$$v_{esc,S} = \sqrt{\frac{2GM}{a}} = \sqrt{\frac{2\left(6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}\right)\left(5.7 \times 10^{26}\,\mathrm{kg}\right)}{2.4 \times 10^8\,\mathrm{m}}} = 18,000\,\mathrm{m/s}, \tag{5}$$

so even when added to the orbital speed of Enceladus (which is $v_{esc,S}/\sqrt{2}$) **(1 point for some mention that you need to change reference frames)**, v is much too small. $\boxed{\text{The material enters into orbit around Saturn.}}$ **(1 point)**

This is the source of Saturn's E ring!

(b) According to Bernoulli's principle, **(2 points)**

$$\frac{1}{2}\rho v^2 + \rho g h + P$$

is a constant. Equating its value for the geyser at the top of the subsurface ocean and the surface of Enceladus, **(1 point)**

$$\frac{1}{2}\rho(0)^2 - \rho g H + P = \frac{1}{2}\rho v^2 + \rho g \cdot 0 + 0, \tag{6}$$

so **(2 points for equation for pressure, 1 point for calculation)**

$$P = \frac{1}{2}\rho v^2 + \rho g H = \left(1000\,\mathrm{kg/m^3}\right)\left(\frac{1}{2}(400\,\mathrm{m/s})^2 + \left(0.11\,\mathrm{m/s^2}\right)\left(25\times 10^3\,\mathrm{m}\right)\right) = \boxed{8.3\times 10^7\,\mathrm{Pa}}.$$

(c) The pressure from the icy shell is **(2 points for equation for pressure, 1 point for calculation)**

$$\rho_i g H = \left(920\,\mathrm{kg/m^3}\right)\left(0.11\,\mathrm{m/s^2}\right)\left(25\times 10^3\,\mathrm{m}\right) = 2.5\times 10^6\,\mathrm{Pa},$$

which is more than an order of magnitude too small.

An alternative calculation of the pressure (either way receives full points) is

$$\frac{4}{3}\pi\left(R^3 - (R-H)^3\right)\rho_i g / \left(4\pi(R-H)^2\right) = 2.8\times 10^6\,\mathrm{Pa}. \tag{7}$$

$\boxed{\text{No, there must be some other source of pressure to accelerate the geysers.}}$

(1 point)

One possibility is that tidal forces from Saturn can lead to cracks in the ice shell and "squeeze" these cracks to accelerate the geyser material. See Kite & Rubin (2016).

6. **(40 points)** *Build your Own H-R Diagram*

The following table contains the apparent visual magnitudes m_v and color indices (B-V) for a sample of stars in a cluster.

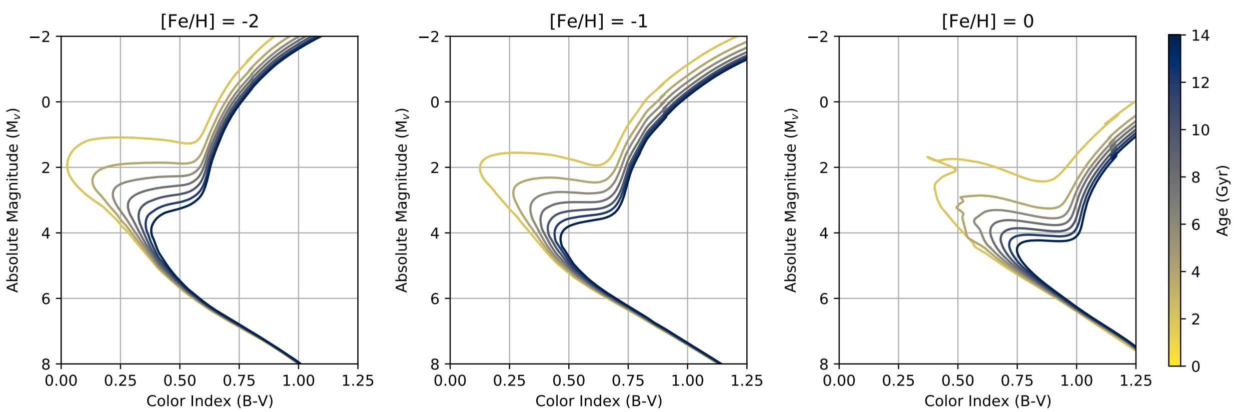
Star #	B-V	m_v	Star #	B-V	m_v	Star #	B-V	m_v	Star #	B-V	m_v
1	0.66	20.0	11	0.80	20.5	21	0.93	21.0	31	0.53	16.5
2	0.45	18.3	12	0.46	16.8	22	0.39	17.0	32	0.43	18.1
3	0.73	19.9	13	0.12	21.1	23	1.01	11.9	33	0.62	19.4
4	0.72	14.2	14	0.20	13.7	24	0.60	19.5	34	0.70	12.3
5	0.38	17.5	15	0.77	20.0	25	-0.03	14.3	35	0.85	13.0
6	0.03	13.9	16	0.51	19.3	26	0.65	15.5	36	0.06	20.9
7	0.55	18.9	17	0.74	13.9	27	0.49	18.9	37	0.61	16.2
8	0.70	14.3	18	0.99	12.0	28	0.61	13.2	38	0.72	20.0
9	0.57	16.3	19	0.53	13.1	29	0.62	15.0	39	0.68	19.7
10	0.91	12.3	20	0.50	16.4	30	0.53	19.0	40	0.11	13.9

For the purposes of this problem, the values have been corrected for extinction and reddening from the interstellar medium.

(a) **(18 points)** Plot the data on an H-R diagram using the provided grid paper. Label at least three of the main features.

Any population of stars that formed together (such as the stars in a cluster) should all be of about the same age and metallicity and should lie along a given curve on the H-R diagram, because their position on the H-R diagram is only a function of their initial mass. These curves are known as *isochrones*.

The following figure shows a selection of modeled isochrones for different ages and metallicities [Fe/H]. The isochrones range in age from 2 Gyr (lightest) to 14 Gyr (darkest) in 2 Gyr steps.



Notes:

- The isochrones plotted here use absolute magnitude M_v on the y-axis, rather than apparent magnitude m_v.
- The isochrones do not include every phase of stellar evolution.
- The data you plotted in part (a) will not perfectly fall along a single curve due to measurement scatter.

(b) **(17 points)** By comparing your H-R diagram to the isochrones, estimate:

  (i) **(5 points)** the metallicity of the cluster (to the nearest integer).

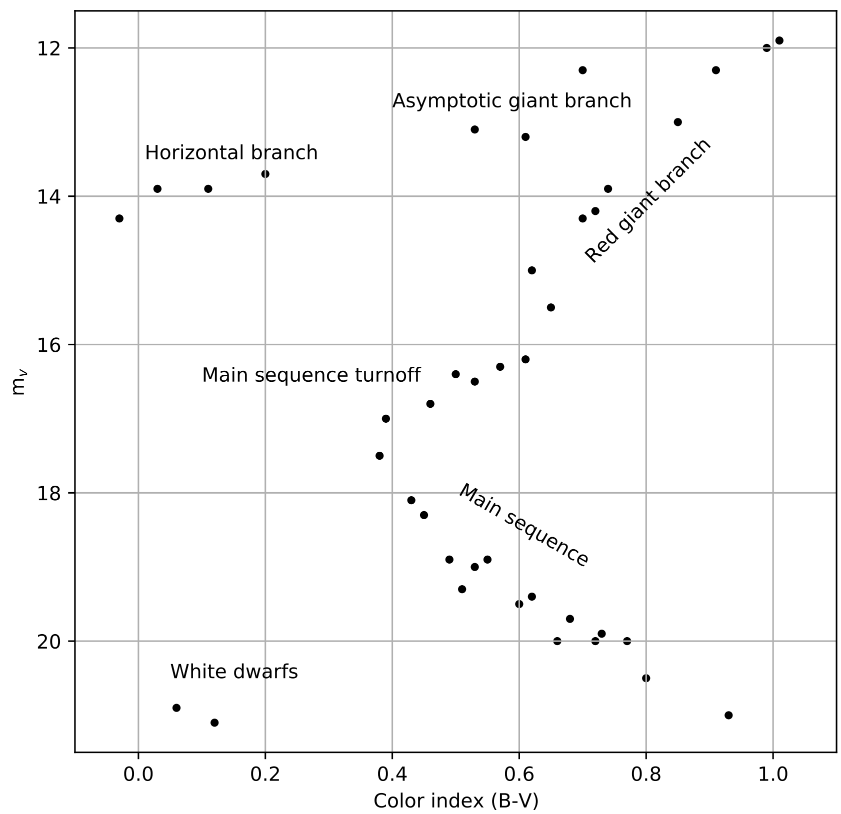
  (ii) **(5 points)** the age of the cluster (to the nearest Gyr).

  (iii) **(7 points)** the distance to the cluster (to the nearest kpc).

(c) **(5 points)** Is this cluster more likely to be an open cluster or a globular cluster? Explain your reasoning.

Solution:

(a) The following scatter plot shows the data on the H-R diagram with several of the main features labeled:



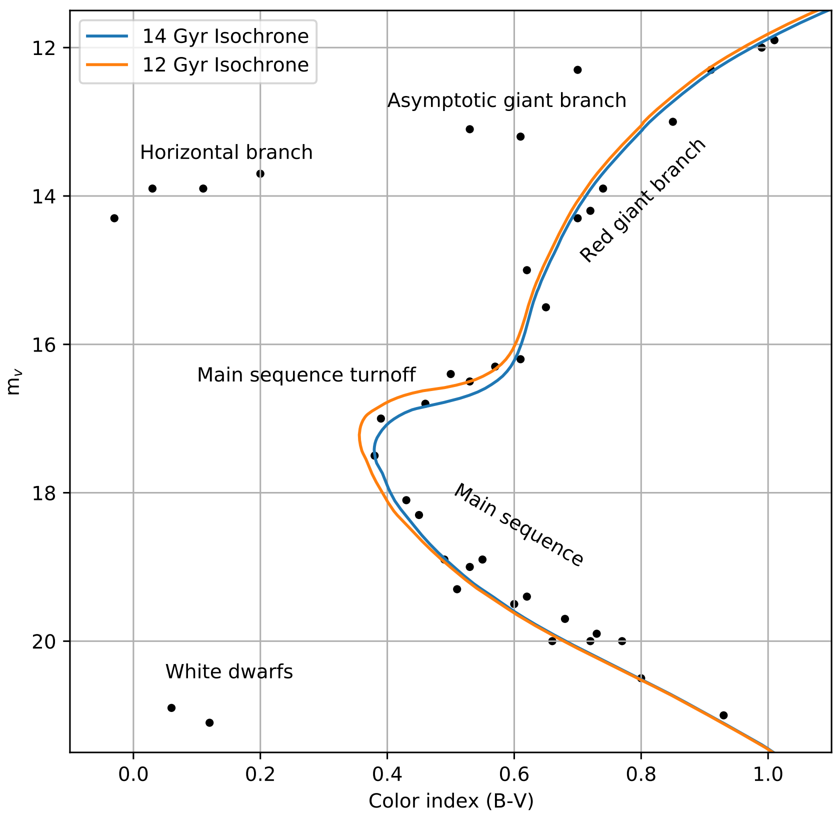
(**9 points** for a correctly drawn plot, **3 points** for labeled axes, and **2 points** per correctly labeled feature up to 3.)

(b) The HR diagram plotted from the given data is a close visual match to isochrones with a metallicity [Fe/H] = -2 and an age of about 12-14 Gyr (formed in the early universe). One easy way to tell is to compare the shape and color index of the main sequence turnoff region.

(**5 points** for [Fe/H] = -2, **5 points** for age from 12-14 Gyr.)

Note that the absolute magnitudes need to be shifted by an overall distance modulus $m_v - M_v$ to match the apparent magnitudes in the given data, so the magnitudes cannot be compared directly. Instead, it's best to compare the color indices as well as the *difference* in magnitude between two points on the isochrone (so that the distance modulus cancels out.)

These two isochrones are plotted on top of the scatter plot for reference (with their magnitudes shifted using the distance modulus described below):



The difference between the apparent and absolute magnitudes is related to the distance to the cluster according to the distance modulus equation:

$$m_v - M_v = 5 \log_{10} \left(\frac{d}{10 \text{ pc}} \right)$$

(**2 points** for applying this equation.)

Visually estimating, we have $m_v - M_v \approx 13.5$ (**2 points**, acceptable range is 13 to 14). This gives $d \approx 5000$ pc (**3 points**, acceptable range is 3900 to 6400 pc).

(c) A metallicity of [Fe/H] = -2 is very low, containing only 1% the metal abundance of the Sun. Both this and an age of 12-14 Gyr are characteristic of globular clusters, rather than open clusters.

(**5 points** for correct answer with explained reasoning.)

7. **(40 points)** *Interstellar Flight*

Kirara is traveling to some stars in the constellation Crux, starting from Earth, and then visiting Gacrux (γ Crucis), Ginan (ε Crucis), Acrux (α Crucis), and Mimosa (β Crucis) in that order. To pass the time, she does some astronomy on each leg of the trip.

(a) **(5 points)** Suppose Kirara starts at rest in the reference frame of the barycenter of the solar system, and that Gacrux, in the same reference frame, has a radial velocity of 20.6 km/s away from Kirara. She has a magic warp drive that can teleport her rocket straight to Gacrux (and preserves her velocity).

However, if she directly teleports, she'll have a large velocity relative to Gacrux once she arrives! She plans to use a normal rocket to first accelerate to a point where she has no velocity relative to Gacrux. If her unfueled rocket weights 1.20×10^5 kg, how much fuel would she need? The exhaust velocity of her rocket is 10.0 km/s.

The Tsiolkovsky rocket equation says that $\Delta v = v_e \ln \frac{m_0}{m_f}$, where Δv is the change in velocity, v_e is the exhaust velocity, m_0 is the initial mass of the rocket (including fuel), and m_f is the final mass of the rocket.

Ignore the proper motion of Gacrux.

(b) **(10 points)** When Kirara gets to Ginan, she takes a break on a (fictional) exoplanet orbiting the star. Ginan has a mass of $1.5\,M_\odot$, and suppose the exoplanet has a mass of $2M_\oplus$, a radius of $1.2R_\oplus$, and a circular orbit with radius 10 AU. She rigs a catapult to give her rocket a one-time boost of Δv in any direction. What is the minimum Δv required for her to escape the combined gravitational field of both Ginan and the exoplanet?

(c) **(5 points)** Kirara arrives at Acrux, which is actually a system of 6 stars. She gets distracted taking measurements of them, and doesn't realize that she's drifting too close to one of the stars! Specifically, she is close to α Crucis Aa, which has a mass of 15.17 $M_\odot$.

She currently is at rest, at a distance R from α Crucis Aa. The free-fall time for an object at distance R to fall into a star with mass 15.17 $M_\odot$ is 24.0 hours, meaning that if Kirara doesn't act, her ship will be pulled into the star in 24 hours!

She repositions to a distance of $2R$ from α Crucis Aa, again with no velocity relative to the star. What is the free-fall time for an object at a distance $2R$ to fall into a star of mass 15.17 $M_\odot$?

Assume α Crucis Aa is at rest, and ignore the gravitational effects of the other stars in the system.

(d) **(20 points)** When Kirara is halfway from Acrux to Mimosa (in linear Euclidean distance), she looks at Mimosa through a telescope. What is the apparent magnitude she observes for Mimosa? The following information will be helpful:

Designation	Name	RA	Dec	Apparent magnitude	Distance to Earth
α Crucis	Acrux	$12^h\ 26^m\ 36^s$	$-63°\ 06'$	0.76	321 ly
β Crucis	Mimosa	$12^h\ 47^m\ 43^s$	$-59°\ 41'$	1.25	352 ly

Assume that Acrux and Mimosa are stationary.

Solution:

(a) **(5 points)** Plugging in the given values directly into the rocket equation, we get

$$20.6 \text{ km/s} = (10.0 \text{ km/s}) \ln \frac{m_0}{1.20 \times 10^5 \text{ kg}}.$$

Then, the total mass of the fueled rocket is

$$(1.20 \times 10^5 \text{ kg}) \exp\left(\frac{20.6 \text{ km/s}}{10.0 \text{ km/s}}\right) = 9.42 \times 10^5 \text{ kg}.$$

Subtracting the mass of the unfueled rocket, we get the mass of the fuel is 9.42×10^5 kg $-$ 1.20×10^5 kg $=$ $\boxed{8.22 \times 10^5 \text{ kg}}$.

(b) **(10 points)** We first compute the gravitational potential at Kirara's location. Assuming her rocket has mass m, the gravitational potential energy from Ginan is

$$-\frac{Gm(1.5M_\odot)}{10 \text{ AU}} = -\frac{6.67 \times 10^{-11} \cdot (1.5 \cdot 1.989 \times 10^{30})}{10 \cdot (1.496 \times 10^{11})} m = -(1.330 \times 10^8)m.$$

Similarly, her gravitational potential energy from the exoplanet is

$$-\frac{Gm(2M_\oplus)}{1.2 \, R_\oplus} = -\frac{6.67 \times 10^{-11} \cdot (2 \cdot 5.976 \times 10^{24})}{1.2 \cdot (6.371 \times 10^6)} m = -(1.043 \times 10^8)m,$$

so the total gravitational potential energy is

$$-(1.330 \times 10^8)m - (1.043 \times 10^8)m = -(2.373 \times 10^8)m.$$

To escape this gravity well, her kinetic energy must equal her negative potential energy, so we have $\frac{1}{2}mv_{total}^2 = (2.373 \times 10^8)m$, so

$$v_{total} = \sqrt{2 \cdot (2.373 \times 10^8)} = 21800 \text{ m/s}.$$

To minimize the Δv needed, Kirara should launch in the same direction as the planet's orbit, so we have $v_{total} = v_{planet} + \Delta v$, so we need to determine v_{planet}.

Since the planet's orbit is circular, we have $M_{planet}\frac{v_{planet}^2}{R} = \frac{GM_{star}M_{planet}}{R^2}$, or

$$v_{planet} = \sqrt{\frac{GM_{star}}{R}} = \sqrt{\frac{6.67 \times 10^{-11} \cdot (1.5 \cdot 1.989 \times 10^{30})}{10 \cdot (1.496 \times 10^{11})}} = 11533 \text{ m/s}.$$

Thus, the required Δv is $(21800 \text{ m/s}) - (11533 \text{ m/s}) = \boxed{10300 \text{ m/s}}$.

(c) **(5 points)** The path of an object starting at rest falling into a star can be thought of as half an "elliptical" orbit with eccentricity 1. By Kepler's 3rd law, the orbital period is proportional to $r^{1.5}$. Thus, doubling the starting distance multiplies the amount of time it takes to fall by $2^{1.5} = 2.83$, so from her new position, she has $24 \cdot 2.83 = \boxed{68 \text{ hours}}$ until she falls into the star.

(d) **(20 points)** We need to determine the distance from Acrux to Mimosa (since Kirara's distance to Mimosa will be half of this distance). To do this, we will first find the angular separation between Acrux and Mimosa using the spherical law of cosines, and then use the triangular law of cosines to find the linear distance between the two stars.

On the celestial sphere, let N be the north celestial pole, A be the position of Acrux, and M be the position of Mimosa. Then,

$$NA = 90^\circ - (-63^\circ 06') = 153.1^\circ,$$

$$NM = 90^\circ - (-59^\circ 41') = 149.7^\circ,$$

and

$$\angle ANM = (12^h\ 47^m\ 43^s) - (12^h\ 26^m\ 36^s) = 21^m\ 7^s = 5.3^\circ.$$

By the spherical law of cosines,

$$\cos(AM) = \cos(NA)\cos(NM) + \sin(NA)\sin(NM)\cos(\angle ANM),$$

so

$$AM = \arccos(\cos(153.1)\cos(149.7) + \sin(153.1)\sin(149.7)\cos(5.3)) = 4.4^\circ.$$

By the triangular law of cosines, using the distances from Earth to Acrux and Mimosa, we have that the distance from Acrux to Mimosa is

$$\sqrt{321^2 + 352^2 - 2 \cdot 321 \cdot 352 \cos(4.4^\circ)} = 41.6 \text{ ly}.$$

Kirara's distance to Mimosa is half of that, which is 20.8 ly.

Now, we calculate the apparent magnitude she observes for Mimosa. The shift in apparent magnitude from being closer to Mimosa is

$$5 \log \frac{\text{distance from Earth to Mimosa}}{\text{Kirara's distance to Mimosa}} = 5 \log_{10} \frac{352}{20.8} = 6.14.$$

Thus, her observed apparent magnitude for Mimosa is $1.25 - 6.14 = \boxed{-4.89}$.

8. **(55 points)** *Watch Out!*

Starting at time $t = 0$, two spaceships A and B begin traveling apart from each other at a relative speed of $v = \beta c$. Let a given ship's reference frame be the frame in which it is constantly at position $x = 0$. For a given event, if it occurs at position x and time t in A's reference frame (call this frame S), let x' and t' be the position and time, respectively, of the event in B's reference frame (call this frame S'). Assume that the spaceships start at the same position, at which point their clocks are initially synchronized.

(a) **(5 points)** Using only Newtonian mechanics, and neglecting all relativistic corrections, write x' and t' in terms of x, t, and v.

(b) **(2 points)** Explain the primary physical contradiction that arises in the above system if β is significantly larger than 0.

Let us now account for relativistic effects. When switching between reference frames moving at a speed v relative to each other, the relativistically correct transformations are

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Here, x, t, x', and t' are defined as before. You may also find the relativistic momentum ($\vec{p} = \gamma m\vec{v}$) and total energy ($E = \gamma mc^2$) formulas helpful.

The spaceships have previously agreed to communicate with each other through light signals. In particular, red light ($\lambda = 700$ nm) indicates that there is imminent danger ahead. Spaceship B, equipped with better radar system, detects a cluster of asteroids along the path that spaceship A is taking and sends a laser signal in red light with a wavelength of λ that lasts a time Δt_0. Both λ and Δt_0 are as measured in B's reference frame, and let the overall emitted power in B's reference frame be P. Use $\beta = 0.4$ for all numeric answers.

You may refer to the following table of color versus *approximate* wavelength for parts (c)(v) and (c)(vi):

λ (nm)	**Color**
$\leqslant 390$	ultraviolet
$390 - 420$	purple
$420 - 480$	blue
$480 - 560$	green
$560 - 585$	yellow
$585 - 610$	orange
$610 - 720$	red
$\geqslant 720$	infrared

(c) **(27 points)** By virtue of relativity, both the observed duration of the signal and the observed wavelength of the signal change. We will now examine this.

(i) **(5 points)** In terms of Δt_0 and $\beta = v/c$, how long does B spend emitting the signal, in A's reference frame? (Denote this as Δt_1.)

(ii) **(7 points)** Due to the finite travel time of light, and given that B is moving away from A, A observes a different length Δt_2 for the signal. *(This is analogous to the non-relativistic Doppler effect.)* Calculate Δt_2 in terms of Δt_0 and β.

(iii) **(3 points)** In terms of λ and β, what wavelength λ' does A observe for the signal?

(iv) **(8 points)** Using your answers to (ii) and (iii), calculate the power P' that A observes for the signal in terms of P and β. (Assume the entire beam emitted by B hits A.)

(v) **(2 points)** Evaluate λ' numerically. What part of the electromagnetic spectrum does this correspond to?

(vi) **(2 points)** In order for A to observe red light at a wavelength λ, numerically evaluate the wavelength λ'' that B would have to emit light at. What part of the electromagnetic spectrum does this correspond to?

An asteroid (of mass $5m$) is traveling directly toward spaceship A (of mass m) at a speed of $0.5c$ in spaceship A's reference frame. From spaceship A's perspective, the asteroid and spaceship B are diametrically opposite each other, and both are moving in the $+x$ direction. Let F_0 be the reference frame in which spaceship A is initially stationary and at the origin. *(Again, use $\beta = 0.4$ for all calculations, and from now on, express all answers in terms of m, c, and numeric prefactors. Onwards, p and E are not referring to four-vectors, rather, p refers to the momentum along the x-axis, E refers to the total energy (rest mass-energy + relativistic kinetic energy), and the "mass" refers to rest mass.)*

(d) **(5 points)** In F_0, what are the combined initial momentum p_i and initial total energy E_i of the asteroid and spaceship A?

Spaceship A and the asteroid undergo a completely inelastic collision, with no mass or energy escaping the system. According to relativity, both p and E are conserved.

(e) **(11 points)** Following the collision, what is the mass M of the resulting object, and what is its velocity v_0 in frame F_0?

(f) **(5 points)** Switching to the reference frame of spaceship B, what are the mass M' and velocity v_0' of the resulting object? Is it moving toward or away from spaceship B?

The relativistic velocity addition formula is given as follows:

$$v' = \frac{v + u}{1 + vu/c^2}$$

where

- v' = velocity of the object in the stationary frame,
- v = velocity of the object in the moving frame,
- u = velocity of the moving frame with respect to the stationary frame.

Solution:

(a) **(5 points)** According to classical Newtonian physics, time is absolute, so the clocks on S and S' stay synchronized **(2 points)**:

$$\boxed{t = t'}.$$

Additionally, we have the following "standard" linear shift for the x-coordinates based on how far ship B has moved **(3 points)**:

$$\boxed{x' = x - vt}.$$

Note that these can also be interpreted as the (fictitious) $c \to \infty$ limits of the Lorentz transformations, thus neglecting all complications that result from near-lightspeed motion.

(b) **(2 points)** The speed of light has to remain constant regardless of the frame of reference. For example, if spaceship A sends a light signal to spaceship B, in B's frame of reference, according to the Newtonian relationship above, the speed of light would be measured as $c - v$, where v is the speed at which spaceship B is moving away from A.

(c) **(28 points)** We have the following:

(i) **(5 points)** We can solve for the inverse Lorentz transformations by inverting the given system of equations:

$$x = \gamma(x' + vt')$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right).$$

In the frame S', the signal begins at $(x', t') = (0, t_0)$ and ends at $(x', t') = (0, t_0 + \Delta t_0)$. Plugging into the above, we see that the change in t is

$$\Delta t_1 = \gamma\left((t_0 + \Delta t_0) + \frac{v}{c^2}(0)\right) - \gamma\left((t_0) + \frac{v}{c^2}(0)\right) = \gamma \Delta t_0,$$

which is just $\boxed{\dfrac{\Delta t_0}{\sqrt{1-\beta^2}}}$. Note that this is just the "standard" time dilation formula.

(ii) **(7 points)** We claim that $\Delta t_2 = (1+\beta)\Delta t_1$ **(5 points)**, just as in the standard non-relativistic Doppler effect. To see this, work in A's frame. Say that B begins emitting the signal at $(x, t) = (vt_1, t_1)$ and finishes emitting the signal at $(x, t) = (v(t_1 + \Delta t_1), t_1 + \Delta t_1)$. Keeping in mind that the signal travels at speed c, the difference in start and end times for receiving the signal is

$$\Delta t_2 = [(t_1 + \Delta t_1) + \frac{1}{c}(v(t_1 + \Delta t_1))] - [t_1 + \frac{1}{c}(vt_1)] = (1+\beta)\Delta t_1,$$

which simplifies to $\boxed{\Delta t_0 \sqrt{\dfrac{1+\beta}{1-\beta}}}$ **(2 points)**.

Note that this step has nothing to do with relativity and is purely a consequence of the finite travel time of light! Additionally, our combined answer should remind you of the relativistic Doppler shift formula; this is because the phases (peaks and troughs) of electromagnetic waves get stretched through exactly the same mechanism.

(iii) **(3 points)** According to the relativistic Doppler effect, the frequency of the light wave measured by spaceship A becomes

$$f = f'\sqrt{\frac{1-\beta}{1+\beta}}$$

Since $\lambda = c/f$,

$$\lambda' = \boxed{\lambda\sqrt{\frac{1+\beta}{1-\beta}}}.$$

(iv) **(8 points)** We combine our results from (ii) and (iii). Specifically, the power decrease is due to a combination of the signal being stretched out in time (resulting in a lower photon collision rate per second with ship A) as well as a redshift of the light itself (resulting in a lower energy per photon received). Since $E = hc/\lambda$ for a photon, we have,

$$\frac{P'}{P} = \frac{\Delta t_0}{\Delta t_2} \cdot \frac{\lambda}{\lambda'} = \left(\sqrt{\frac{1-\beta}{1+\beta}}\right)^2 = \frac{1-\beta}{1+\beta} \to P' = \boxed{P \cdot \frac{1-\beta}{1+\beta}}.$$

(**2.5 points** for accounting for $\frac{P'}{P} = \frac{\Delta t_0}{\Delta t_2}$. **2.5 points** for accounting for $\frac{P'}{P} = \frac{\lambda}{\lambda'}$. **1 point** for combining to form $\frac{P'}{P} = \frac{\Delta t_0}{\Delta t_2} \cdot \frac{\lambda}{\lambda'}$. **2 points** for the correct final answer.)

(v) **(2 points)** We calculate **(1 point)**

$$\lambda' = (700\,\text{nm})\sqrt{\frac{1+0.4}{1-0.4}} \approx \boxed{1070\,\text{nm}}.$$

Spaceship A doesn't see the red light, as this is in the **(1 point)** $\boxed{\text{infrared}}$ part of the spectrum.

(vi) **(2 points)** Again, according to the Doppler shift formula,

$$\lambda = \lambda''\sqrt{\frac{1+\beta}{1-\beta}},$$

giving **(1 point)**

$$\lambda'' = (700\,\text{nm})\sqrt{\frac{1-0.4}{1+0.4}} \approx \boxed{460\,\text{nm}},$$

which is **(1 point)** $\boxed{\text{blue}}$ light.

(d) **(5 points)** In spaceship A's frame of reference, $v_{\text{asteroid}} = 0.5c$, and the spaceship is at rest. Therefore, the total initial momentum is **(2 points)**

$$P_i = \gamma_{\text{asteroid}} \cdot 5m \cdot (0.5c) = \boxed{\frac{5}{\sqrt{3}}mc}.$$

In spaceship A's frame of reference, $E_{\text{spaceship A}} = mc^2$ and

$$E_{\text{asteroid}} = \gamma_{\text{asteroid}}(5m)(c^2) = \frac{10}{\sqrt{3}}mc^2.$$

Therefore, **(3 points)**

$$E_i = E_{\text{spaceship A}} + E_{\text{asteroid}} = \boxed{\left(1+\frac{10}{\sqrt{3}}\right)mc^2}.$$

(e) **(11 points)** We assume that in spaceship A's frame of reference, spaceship B moves at $+0.4c$ relative to the spaceship A. *(From now on, algebraic expressions for the coefficients start to get messy, so we begin to carry coefficients numerically.)*

For the resulting mass M, Lorentz factor γ, and final velocity v_0, according to conservation of momentum,

$$M\gamma v_0 = \frac{5}{\sqrt{3}}mc,$$

and according to conservation of energy,

$$M\gamma c^2 = mc^2\left(1+\frac{10}{\sqrt{3}}\right),$$

Dividing the momentum equation by the energy equation gives:

$$\frac{P}{E} = \frac{v_0}{c^2} = \frac{\frac{5}{\sqrt{3}}mc}{mc^2\left(1+\frac{10}{\sqrt{3}}\right)}.$$

(**3 points** for using $\frac{P}{E} = \frac{v_0}{c^2}$. **4 points** for correct v_0.)

Simplifying,

$$v_0 \approx \boxed{0.426c}.$$

Note that both algebraic and numerical answers for these coefficients are acceptable.

From here, we can calculate

$$M = \frac{E\sqrt{1-(v_0/c)^2}}{c^2} \approx \frac{mc^2\left(1+\frac{10}{\sqrt{3}}\right)\cdot\sqrt{1-0.426^2}}{c^2} \approx \boxed{6.13m}.$$

(**3 points** for using $M = \frac{E\sqrt{1-(v_0/c)^2}}{c^2}$. **1 point** for $M \approx 6.13m$.)

(f) **(5 points)** Trivially, $M' = M \approx \boxed{6.13m}$ **(2 points)** by basic relativistic principles, so we focus on computing v_0' via the relativistic velocity addition formula. In the frame of spaceship B, the observed combined velocity is

$$v_0' = \frac{v_0 + v_A}{1 + \frac{v_0 v_A}{c^2}}$$

where $v_A = -0.4c$ and $v_0 = 0.4262c$ (we carry more decimal places in the solution for clarity). Substituting the values gives **(2 points)**

$$v_0' \approx \boxed{0.0316c}.$$

The combined mass of the asteroid and spaceship A is slowly traveling $\boxed{\text{toward}}$ spaceship B in its frame of reference **(1 point)**.

9. **(85 points)** *Neutron Star Dynamics!*
*Note that the provided **Answer Sheet** for this question is only for subtasks (c)(vi) and (d)(v). Do all other work on blank pages.*
In this question, we do a detailed dive into several elements of neutron star dynamics and energy losses. While understanding the internal nature and makeup of a neutron star requires quantum chromodynamics and general relativity, other properties, such as rotation rate, are far more elementary to model. Over time, a neutron star's rotation rate slowly decreases due to factors such as energy loss and mass accretion. By measuring changes in a neutron star's rotation rate, astronomers can make inferences about the physics surrounding a neutron star and its environment. Neglect more complex behavior (such as QED/QCD/GR corrections), instead assuming Newtonian mechanics holds unless otherwise stated.

Let $\Omega(t)$ be the star's angular velocity around its axis as a function of time. In several scenarios, we can observe a proportionality relation of the form $\dot{\Omega} \propto -\Omega^n$, where n is the neutron star's **braking index**. *(Throughout this problem, dots denote time derivatives, i.e. $df/dt = \dot{f}$ and $d^2f/dt^2 = \ddot{f}$.)*

(a) **(10 points)** We will begin by examining the basic behavior of the time evolution of Ω.

(i) **(5 points)** Express n in terms of Ω, $\dot{\Omega}$, and $\ddot{\Omega}$.

(ii) **(3 points)** If $n > 1$, then we can deduce that $\Omega(t) \propto t^{-\alpha}$ for some α. Compute α in terms of n.

(iii) **(2 points)** What type of curve does $\Omega(t)$ versus t trace if $n = 1$?

This relation allows astronomers to measure n for a given star. We will now examine several mechanisms for rotational slowdown and the subsequent values for n, treat each of these mechanisms as **independent** of each other unless otherwise stated. (In other words, do not combine mechanisms until part (d)(v).)

(b) **(8 points)** First, we will look at a primitive model for mass accretion. Let M be the neutron star's mass and R be its radius. Assume the rate of mass accretion $\dot{M}$ is constant and that the infalling matter has no initial angular momentum relative to the rotation axis. Additionally, assume (slightly unrealistically) that the neutron star is a uniformly dense sphere of matter.

(i) **(2 points)** Write down the expression for the angular momentum of the neutron star L in terms of M, R, and Ω, up to a numerical constant. Use the fact that L stays constant for problems (ii) and (iii) of this subpart.

(ii) **(3 points)** Now, assume that as the star accretes mass, the density ρ stays constant over time. Calculate the value of n.

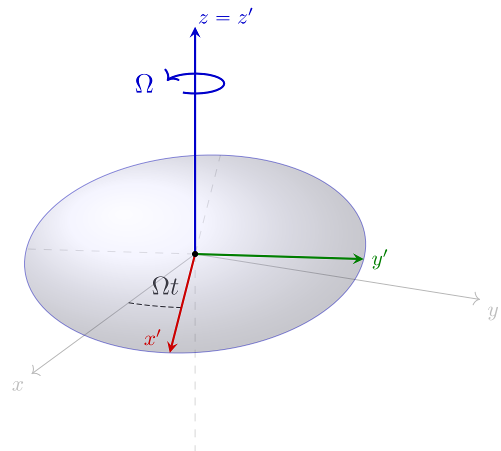
(iii) **(3 points)** Frontier QCD simulations currently appear to imply that, in contrast to the above, R stays relatively constant as M changes. Treating this new relationship as exact, calculate the new value of n.

(c) **(40 points)** Next, we will examine gravitational wave emission from the neutron star. To leading order, gravitational wave emission is caused by oscillations in the *mass quadrupole moment*, which is the deviation from spherical symmetry when projected onto an axis, of the neutron star. The general formula for the traceless quadrupole moment terms is

$$Q_{ij} = \int \rho(\mathbf{r})(r_i r_j - \frac{1}{3}r^2\delta_{ij})d^3\mathbf{r},$$

where $\mathbf{r}$ is the displacement from the center of the ellipsoid, $i, j \in \{x, y, z\}$ are coordinate axes, r_i are projections of $\mathbf{r}$ onto these axes (so $r_x = x$), and $\delta_{ij} = 1$ for $i = j$ with $\delta_{ij} = 0$ otherwise. **Such integrals are over all space** unless otherwise specified. *(Formally, the quadrupole moment is a 3 by 3 matrix Q with 9 terms Q_{ij}, but you will not need any knowledge of linear algebra for this problem.)*

We model the neutron star as a rotating ellipsoid (which deviates very slightly from a perfect sphere).



We establish a fixed xyz coordinate system for reference, which the neutron star rotates at angular speed Ω relative to. The neutron star has body symmetry axes x', y', and $z' = z$, as per the diagram. By the definition of body symmetry axes,

$$\int \rho(\mathbf{r}')(r'_i r'_j) d^3\mathbf{r}' = I_i \delta_{ij}.$$

In other words, the moments of inertia around the body axes x', y', and z' are I_x, I_y, and I_z, respectively, and all "cross" terms between different axes vanish. (Let $I_x > I_y$.) Some elementary trigonometry yields

$$\begin{cases} x = x'\cos(\Omega t) - y'\sin(\Omega t), \\ y = x'\sin(\Omega t) + y'\cos(\Omega t), \\ z = z'. \end{cases}$$

We will now compute the time evolution of Q_{ij}. Assume Ω is constant until part (v).

(i) **(12 points)** Using the above coordinate transformations and inertia integrals, calculate all terms Q_{ij}, expressing your answer in terms of the body axis moments of inertia and trigonometric functions of Ωt. For example,

$$Q_{xz} = \int \rho(\mathbf{r})(xz - \frac{1}{3}r^2\delta_{xz})d^3\mathbf{r} = \int \rho(\mathbf{r}')(xz)d^3\mathbf{r}' = \int \rho(\mathbf{r}')(x'\cos(\Omega t) - y'\sin(\Omega t))z'd^3\mathbf{r}',$$

$$= \cos(\Omega t)\int \rho(\mathbf{r}')x'z'd^3\mathbf{r}' - \sin(\Omega t)\int \rho(\mathbf{r}')y'z'd^3\mathbf{r}' = 0,$$

since both integrals vanish as stated.

(ii) **(7 points)** For the first step of our calculations, we will need the second time derivative $\frac{d^2Q_{ij}}{dt^2}$ of the quadrupole moment. Calculate these terms and indicate which ones are nonzero. *(You may find one or more of the identities* $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, $\sin 2\alpha = 2\sin\alpha\cos\alpha$, *and* $\cos^2\alpha + \sin^2\alpha = 1$ *helpful.)*

Detecting gravitational waves involves measuring the *strain* h, or the fractional change in length $\Delta L/L$ of spacetime caused by the gravitational wave. For the purposes of this problem, we ignore nuances such as viewing angle and GW polarization regimes. As such, we approximate the behavior as

$$h_{ij}(t) = \frac{2G}{c^4 d}\frac{d^2Q_{ij}}{dt^2},$$

where d is the distance from the observer to the neutron star. *(Note that there is a time offset due to the nonzero time it takes gravitational waves to reach the observer, but we ignore this, as it is not highly relevant in understanding the qualitative behavior of the system.)*

(iii) **(3 points)** For convenience, we only look at one term of h_{ij} to capture the underlying behavior. Calculate the amplitude h_0 and angular frequency Ω_{GW} of the oscillations in h_{xx}. Express your answers in terms of I_z, $\epsilon = \frac{I_x - I_y}{I_z}$, Ω, and fundamental constants.

(iv) **(6 points)** According to general relativity, the radiated power is given by

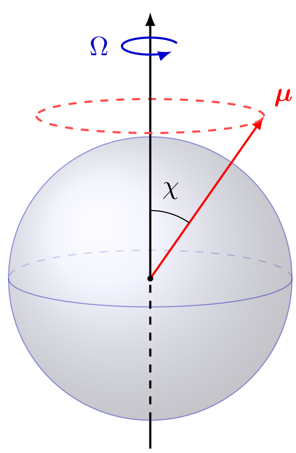
$$P = \frac{G}{5c^5} \sum_{ij} \left(\frac{d^3 Q_{ij}}{dt^3} \right)^2,$$

where the sum runs over all 9 pairs of indices $i, j \in \{x, y, z\}$. Calculate P in terms of I_z, ϵ, Ω, and fundamental constants.

(v) **(6 points)** The power for the gravitational waves comes purely from the rotational kinetic energy $E = \frac{1}{2} I_z \Omega^2$ of the neutron star. Assuming its mass, radius, and inertia moments stay constant, calculate the value of $\dot{\Omega}$ in terms of Ω, I_z, ϵ, and fundamental constants. From this, extract the value of n.

(vi) **(6 points)** As the neutron star slowly loses rotational kinetic energy while keeping its inertia moments constant, oscillations of the observed gravitational wave strain h change in both period and amplitude. On your **Answer Sheet**, **sketch** a graph for $h_{xx}(t)$ versus time t that reflects these changes. *(Exaggerate your graph so that significant visible changes in both period and amplitude occur over the course of several periods. Your graph need not be to scale.)*

(d) **(27 points)** Finally, we will consider magnetic dipole radiation, which is electromagnetic radiation induced by the rotation of the star and subsequent movement of its magnetic field. This is the main source of energy loss in pulsar systems. Let the magnetic dipole moment be μ with an inclination χ to the rotation axis (here, the z-axis). Take $\chi \in (0, \frac{\pi}{2})$.



Some simple geometry gives

$$\boldsymbol{\mu}(t) = \mu(\sin\chi\cos(\Omega t), \sin\chi\sin(\Omega t), \cos\chi)$$

in component form.

(i) **(4 points)** Calculate the second derivative $\ddot{\boldsymbol{\mu}}(t)$ in component form. Using this, evaluate its magnitude $|\ddot{\boldsymbol{\mu}}|$.

(ii) **(5 points)** According to the Larmor formula, the power radiated by the magnetic dipole is given by

$$P = \frac{\mu_0}{6\pi c^3} |\ddot{\boldsymbol{\mu}}|^2,$$

where μ_0 is the permeability of free space. As in the previous section, assume that this power comes from a change in the rotational kinetic energy $E = \frac{1}{2} I_z \Omega^2$, where we again take I_z as the (constant) moment of inertia around the rotation axis. Calculate $\dot{\Omega}$ as a function of Ω and any other relevant variables. Using this, calculate the value of n. (Refer to this value as n_0.)

(iii) **(9 points)** Nevertheless, neutron stars are sometimes observed to have n-values that significantly differ from n_0. We will now model this. Specifically, take your same $P = -\dot{E}$ expression as before, but now assume that μ and χ are no longer fixed, instead potentially having small nonzero rates of change. Using your result from part (a)(i), calculate a new value for n. Express your answer in terms of Ω, $\dot{\Omega}$, μ, $\dot{\mu}$, χ, $\dot{\chi}$, and fundamental constants. Again, assume that the mass, radius, and composition of the neutron star stay constant.

(iv) **(2 points)** Assume that the neutron star's magnetic moment decays exponentially according to

$$\mu(t) \propto e^{-t/\tau_B}.$$

Rewrite your expression for n from part (iii) in terms of Ω, $\dot{\Omega}$, τ_B, χ, $\dot{\chi}$, and fundamental constants.

(v) **(7 points)** The Crab Pulsar is observed to have a long-term average braking index of $n_{obs} = 2.51 \pm 0.01$. For each of the following, mark on your **Answer Sheet** whether it would cause n to deviate **toward** n_{obs}, **away from** n_{obs}, or have **no change**, assuming a default value of $n = n_0$ (remember $\dot{\Omega} < 0$ and $\chi \in (0, \frac{\pi}{2})$):

(A) A gradual increase in the inclination χ,

(B) A gradual decrease in the internal magnetic dipole moment μ,

(C) Gradual accretion of a nearby star (use your result from part (b)(iii) of the problem).

(D) Significant gravitational wave emission (use your result from part (c) of the problem).

Solution: *Note that in the below solutions, partial credit point values are indicated for certain key steps. Correct answers with reasonable work earn full credit as usual.*

(a) **(10 points)**

(i) **(5 points)** Let $\dot{\Omega} = -k\Omega^n \rightarrow \ddot{\Omega} = -nk\Omega^{n-1}$ **(2 points)** for some constant k. Then $\frac{\ddot{\Omega}}{\dot{\Omega}} = n\frac{\dot{\Omega}}{\Omega} \rightarrow n = \boxed{\frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2}}$. (This is how n is computed in practice.)

(ii) **(3 points)** We get

$$\dot{\Omega} \propto t^{-1-\alpha} \propto \Omega^n \propto t^{-n\alpha} \rightarrow 1 + \alpha = n\alpha \rightarrow \alpha = \boxed{\frac{1}{n-1}}.$$

(**1 point** for each step above).

(iii) **(2 points)** Notice that the above relation diverges if $n = 1$. Instead,

$$\dot{\Omega} \propto -\Omega \leftrightarrow \dot{\Omega} = -k\Omega \leftrightarrow \Omega(t) = \Omega_0 \exp(-kt),$$

so Ω is $\boxed{\text{decreasing exponentially}}$.

(b) **(8 points)**

(i) **(2 points)** We just have $L = I\Omega$ and $I \propto MR^2$ up to a constant **(1 point)**. Thus $L \propto \boxed{\Omega MR^2}$. For the rest of part (b), we solely carry proportionalities in this solution for convenience.

(ii) **(3 points)** Since L is constant, $\Omega \propto \frac{1}{MR^2}$ **(0.5 points)** and $M \propto \rho R^3$ **(1 point)**, meaning $\Omega \propto M^{-\frac{5}{3}}$ **(0.5 points)**. Using $\dot{M} = \textit{const}$, we can now calculate n using the formula from part (a), or alternatively we can scale the time axis so that $M = \alpha t$ for some constant α, giving $\Omega \propto t^{-\frac{5}{3}} \rightarrow \dot{\Omega} \propto t^{-\frac{8}{3}} \propto \Omega^{\frac{8}{5}}$, from which we can read off $n = \boxed{\frac{8}{5}}$ **(1 point)**.

(iii) **(3 points)** We write $\Omega \propto M^{-1}$ **(1 point)** and proceed as before, giving $\Omega \propto t^{-1} \to \dot{\Omega} \propto t^{-2} \propto \Omega^2$, from which we can read off $n = \boxed{2}$.

(c) **(40 points)**

(i) **(12 points)** Notice that $Q_{ij} = Q_{ji}$ since the formula is symmetric. As given in the problem, we know that $Q_{xz} = Q_{zx} = \boxed{0}$. Analogous logic (or symmetry) also yields $Q_{yz} = Q_{zy} = \boxed{0}$. Now, all we need to do is compute Q_{xx}, Q_{yy}, Q_{zz}, and $Q_{xy} = Q_{yx}$. We go in the above order for clarity (note that since we integrate over all space, we can trivially switch $d^3\mathbf{r}$ and $d^3\mathbf{r}'$ **(1 point)**, and recall $r^2 = x^2 + y^2 + z^2 = (r')^2 = (x')^2 + (y')^2 + (z')^2$ **(1 point)**):

$$Q_{xx} = \int \rho(\mathbf{r})(x^2 - \frac{1}{3}r^2)d^3\mathbf{r} = \int \rho(\mathbf{r})(x^2 - \frac{1}{3}r^2)d^3\mathbf{r}'$$

$$= \int \rho(\mathbf{r}) \left((x' \cos(\Omega t) - y' \sin(\Omega t))^2 - \frac{1}{3}(x')^2 - \frac{1}{3}(y')^2 - \frac{1}{3}(z')^2 \right) d^3\mathbf{r}'$$

$$= \int \rho(\mathbf{r})(x')^2 \left(\cos^2(\Omega t) - \frac{1}{3} \right) d^3\mathbf{r}' + \int \rho(\mathbf{r})(y')^2 \left(\sin^2(\Omega t) - \frac{1}{3} \right) d^3\mathbf{r}'$$

$$+ \int \rho(\mathbf{r})x'y' \left(-2\sin(\Omega t)\cos(\Omega t) \right) d^3\mathbf{r}' + \int \rho(\mathbf{r})(z')^2 \left(-\frac{1}{3} \right) d^3\mathbf{r}'$$

$$= I_x \left(\cos^2(\Omega t) - \frac{1}{3} \right) + I_y \left(\sin^2(\Omega t) - \frac{1}{3} \right) + 0 \left(-2\sin(\Omega t)\cos(\Omega t) \right) + I_z \left(-\frac{1}{3} \right),$$

giving

$$Q_{xx} = \boxed{I_x \cos^2(\Omega t) + I_y \sin^2(\Omega t) - \frac{I_x + I_y + I_z}{3}}.$$

Switching the x' expression for the y' expression and using analogous logic yields

$$Q_{yy} = \boxed{I_y \cos^2(\Omega t) + I_x \sin^2(\Omega t) - \frac{I_x + I_y + I_z}{3}}.$$

Next, we get

$$Q_{zz} = \int \rho(\mathbf{r})(z^2 - \frac{1}{3}r^2)d^3\mathbf{r} = \int \rho(\mathbf{r})(\frac{2}{3}(z')^2 - \frac{1}{3}(x')^2 - \frac{1}{3}(y')^2)d^3\mathbf{r} = \boxed{\frac{2I_z - I_x - I_y}{3}},$$

and

$$Q_{xy} = \int \rho(\mathbf{r}')(x' \cos(\Omega t) - y' \sin(\Omega t))(x' \sin(\Omega t) + y' \cos(\Omega t))d^3\mathbf{r}'$$

$$= \int \rho(\mathbf{r})(x')^2 \left(\sin(\Omega t)\cos(\Omega t) \right) d^3\mathbf{r}' + \int \rho(\mathbf{r})(y')^2 \left(-\sin(\Omega t)\cos(\Omega t) \right) d^3\mathbf{r}'$$

$$+ \int \rho(\mathbf{r})x'y' \left(\cos^2(\Omega t) - \sin^2(\Omega t) \right) d^3\mathbf{r}',$$

thus

$$Q_{xy} = Q_{yx} = \boxed{(I_x - I_y)\sin(\Omega t)\cos(\Omega t)},$$

finishing.

(***4 points*** *for interpreting the* δ_{ij} *terms and substituting the* $x'y'z'$ *system accordingly,* ***0.25 points*** *for each* $Q_{ij} = 0$ *term,* ***1 point*** *for each distinct nonzero* Q_{ij}*, and* ***1 point*** *for getting everything correct.*)

(ii) **(7 points)** Trivially, Q_{xz}, Q_{yz}, Q_{zx}, Q_{zy} (**1 point**), and Q_{zz} **(0.5 points)** are all constant, so we only need to compute three distinct time derivatives. Notice that we can rewrite

$$Q_{xx} = I_x \cos^2(\Omega t) + I_y \sin^2(\Omega t) - \frac{I_x + I_y + I_z}{3}$$

$$= (I_x - I_y)\frac{\cos^2(\Omega t) - \sin^2(\Omega t)}{2} + (I_x + I_y)\frac{\cos^2(\Omega t) + \sin^2(\Omega t)}{2} - \frac{I_x + I_y + I_z}{3}.$$

Now, since $\cos^2(\Omega t) + \sin^2(\Omega t)$ is constant and $\cos^2(\Omega t) - \sin^2(\Omega t) = \cos(2\Omega t)$, and the second derivative of this is just $-4\Omega^2\cos(2\Omega t)$, we can differentiate the time-varying term $\frac{I_x - I_y}{2}\cos(2\Omega t)$ twice to get,

$$\ddot{Q}_{xx} = \boxed{-2\Omega^2(I_x - I_y)\cos(2\Omega t)}.$$

Analogously, switching I_x and I_y,

$$\ddot{Q}_{yy} = \boxed{2\Omega^2(I_x - I_y)\cos(2\Omega t)}.$$

Next, notice that $Q_{xy} = (I_x - I_y)\sin(\Omega t)\cos(\Omega t) = \frac{1}{2}(I_x - I_y)\sin(2\Omega t)$, which differentiates twice to give,

$$\ddot{Q}_{xy} = \ddot{Q}_{yx} = \boxed{-2\Omega^2(I_x - I_y)\sin(2\Omega t)}.$$

Since all other terms are constant, their second time derivatives are zero, so we are done. *(**1.5 points** for each correct boxed expression. **1 point** extra for getting everything correct.)*

(iii) **(3 points)** We substitute to get **(1 point)**

$$h_{xx}(t) = \frac{-4\Omega^2 G}{c^4 d}(I_x - I_y)\sin(2\Omega t),$$

which has amplitude $h_0 = \frac{4\Omega^2 G}{c^4 d}(I_x - I_y) = \boxed{\frac{4\Omega^2 G}{c^4 d}I_z\epsilon}$ **(1 point)** and angular frequency $\Omega_{GW} = \boxed{2\Omega}$ **(1 point)**.

(iv) **(6 points)** Using our results from (ii), the only terms (**0.5 points** for correctly ignoring all others) with nonzero third derivative are **(0.5 points each)**

$$\dddot{Q}_{xx} = -\dddot{Q}_{yy} = 4\Omega^3(I_x - I_y)\sin(2\Omega t)$$

and **(0.5 points)**

$$\dddot{Q}_{xy} = \dddot{Q}_{yx} = -4\Omega^3(I_x - I_y)\cos(2\Omega t),$$

thus giving

$$P = \frac{G}{5c^5}\left(2\left(4\Omega^3(I_x - I_y)\sin(2\Omega t)\right)^2 + 2\left(4\Omega^3(I_x - I_y)\cos(2\Omega t)\right)^2\right)$$

$$= \frac{32G}{5c^5}\Omega^6(I_x - I_y)^2(\sin^2(2\Omega t) + \cos^2(2\Omega t)) = \boxed{\frac{32G}{5c^5}\Omega^6 I_z^2\epsilon^2}.$$

*(**1.5 points** for any equivalent P, and **1.5 points** extra for a time-independent P.)*

(v) **(6 points)** We know

$$P = -\dot{E} = -I_z\Omega\dot{\Omega} = \frac{32G}{5c^5}\Omega^6 I_z^2 \epsilon^2,$$

*(**2 points** for calculating $\dot{E}$ and **1 point** for substituting correctly)* yielding **(1.5 points)**

$$\dot{\Omega} = \boxed{-\frac{32G}{5c^5}\Omega^5 I_z \epsilon^2},$$

and so $n = \boxed{5}$ **(1.5 points)**.

(vi) **(6 points)** Recall from part (iii) that $h_0 \propto \Omega^2$ and $\Omega_{GW} \propto \Omega$. As the star loses rotational kinetic energy $E = \frac{1}{2}I_z\Omega^2$, since I_z stays constant, Ω must decrease. From this, we see that both h_0 **(1.5 points)** and Ω_{GW} decrease **(1.5 points)**, meaning that our graph of $h_{xx}(t)$ decreases in amplitude and increases in period. We get a graph of the following form (note that, as requested, the changes are exaggerated relative to the true time scale of the system):

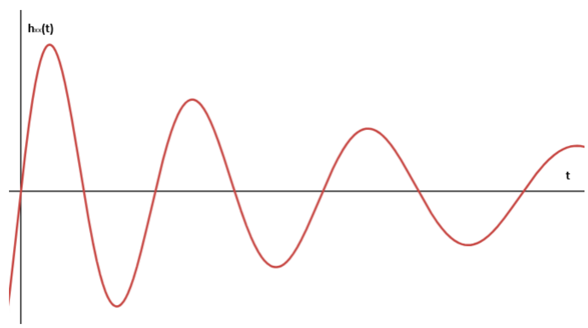


Figure 2: GW Signal Evolution

*(**2 points** for a sinusoid-style shape and **1 point** for everything correct.)*

(d) **(27 points)**

(i) **(4 points)** We can calculate

$$\ddot{\boldsymbol{\mu}}(t) = \boxed{\mu(-\Omega^2 \sin\chi \cos(\Omega t), -\Omega^2 \sin\chi \sin(\Omega t), 0)},$$

which has magnitude

$$|\ddot{\boldsymbol{\mu}}| = \sqrt{(-\mu\Omega^2 \sin\chi \cos(\Omega t))^2 + (-\mu\Omega^2 \sin\chi \sin(\Omega t))^2} = \boxed{\mu\Omega^2 \sin\chi}.$$

*(**1 point** per component and **1 point** for the magnitude.)*

(ii) **(5 points)** We substitute **(4 points)**

$$P = -\dot{E} = -I_z\Omega\dot{\Omega} = \frac{\mu_0}{6\pi c^3}(\mu\Omega^2 \sin\chi)^2 \to \dot{\Omega} = \boxed{-\frac{\mu_0}{6\pi c^3 I_z}\mu^2\Omega^3 \sin^2\chi},$$

from which we read off $n_0 = \boxed{3}$ **(1 point)**.

(iii) **(9 points)** We want $n = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2}$, so we start by computing $\ddot{\Omega}$: **(3 points)**

$$\ddot{\Omega} = \frac{d}{dt}\left[-\frac{\mu_0}{6\pi c^3 I_z}\mu^2\Omega^3\sin^2\chi\right] = -\frac{\mu_0}{6\pi c^3 I_z}\frac{d}{dt}\left[\mu^2\Omega^3\sin^2\chi\right],$$

and so, using the product rule, we calculate:

$$\frac{d}{dt}\left[\mu^2\Omega^3\sin^2\chi\right] = \mu^2\Omega^3\sin^2\chi\left(\frac{2\dot{\mu}}{\mu} + \frac{3\dot{\Omega}}{\Omega} + \frac{2\dot{\chi}\cos\chi}{\sin\chi}\right),$$

which yields **(3 points)**

$$\ddot{\Omega} = -\frac{\mu_0}{6\pi c^3 I_z}\mu^2\Omega^3\sin^2\chi\left(\frac{2\dot{\mu}}{\mu} + \frac{3\dot{\Omega}}{\Omega} + 2\dot{\chi}\cot\chi\right),$$

thus giving **(3 points)**

$$n = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} = \frac{\Omega}{\dot{\Omega}}\cdot\frac{-\frac{\mu_0}{6\pi c^3 I_z}\mu^2\Omega^3\sin^2\chi\left(\frac{2\dot{\mu}}{\mu} + \frac{3\dot{\Omega}}{\Omega} + 2\dot{\chi}\cot\chi\right)}{-\frac{\mu_0}{6\pi c^3 I_z}\mu^2\Omega^3\sin^2\chi}$$

$$= \boxed{3 + \frac{2\Omega}{\dot{\Omega}}\left(\frac{\dot{\mu}}{\mu} + \dot{\chi}\cot\chi\right)}$$

as desired. Note that the problem statement asks us to express our answer in terms of $\dot{\Omega}$ and not I_z, so we do not substitute further. Additionally, setting $\dot{\mu} = 0$ and $\dot{\chi} = 0$ recovers $n_0 = 3$, which is a helpful sanity check for our result.

(iv) **(2 points)** This translates to $\frac{\dot{\mu}}{\mu} = -\frac{1}{\tau_B}$ **(1 point)**, or

$$n = \boxed{3 + \frac{2\Omega}{\dot{\Omega}}\left(\dot{\chi}\cot\chi - \frac{1}{\tau_B}\right)}.$$

(v) **(7 points)** We have $n_{obs} < n_0$. For choices (A) and (B), deviating toward n_0 manifests as having $\frac{2\Omega}{\dot{\Omega}}\left(\frac{\dot{\mu}}{\mu} + \dot{\chi}\cot\chi\right) < 0$. Since $\dot{\Omega} < 0$, this is equivalent to $\left(\frac{\dot{\mu}}{\mu} + \dot{\chi}\cot\chi\right) > 0$. This is true if $\dot{\chi} > 0$ and false if $\dot{\mu} > 0$, so (A) causes it to deviate **toward** n_{obs} and (B) causes it to deviate **away** from n_{obs}.

For (C), remember that pure accretion causes a value of $n = 2$, which is in the same direction of n_0 as n_{obs}. Evidently, since there are other relevant factors, n will only go "part" of the way to 2, but this is still deviation **toward** n_{obs}.

For (D), we have that pure GW emission causes a value of $n = 5$, which is in the opposite direction of n_{obs} from n_0. This causes GW emission to make the neutron star deviate **away** from n_{obs}.

	A	B	C	D
Toward n_{obs}	x		x	
Away From n_{obs}		x		x
No Effect				

Table 1: Final Answers

*(**1.5 points** per correct column, **1 point** for getting everything correct.)*