USAAAO 2025 - First Round

February 8th, 2025

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1. Imagine you are on Deimos right now and you want to escape Mars because you are bored. If Deimos is currently at 23,460 km away from the center of Mars and its speed is currently around 1.35 km/s, how much more speed do you and Deimos need to escape Mars? *The mass of Mars is* 6.39×10^{23} *kg.*

   (a) 557 m/s
   (b) 427 m/s
   (c) 377 m/s
   (d) 207 m/s
   (e) None

Solution: To escape Mars, you would need to know the escape velocity of Deimos at its current position.

$$\begin{aligned} v_{escape} &= \sqrt{\frac{2GM}{R}} \\ &= \sqrt{\frac{2 \times 6.673 \times 10^{-11} \times 6.39 \times 10^{23}}{23460 \times 10^3}} \\ &\approx 1.907 \text{ km/s} \end{aligned}$$

Hence, you would need

$$\begin{aligned} \Delta v &\approx 1,907 - 1,35 \\ &\approx 557 \text{ m/s} \end{aligned}$$

$\boxed{\textbf{Answer: A}}$

2. **Use the following information for the next two questions.** Suppose a spherical gas giant has radius R, temperature T, uniform density and composition, and fixed volumetric heat capacity. Assuming heat leaves solely through blackbody radiation, calculate how the planet's cooling rate $-dT/dt$ depends on R, T.

   (a) T^2/R
   (b) T^4/R
   (c) T^4/R^3

(d) T^2R

(e) T^4R^3

Solution: The internal energy of the gas giant is $U \sim R^3T$ since its volume $V \sim R^3$, while its heat escapes at a rate given by the Stefan-Boltzmann law, $P = -dU/dt \sim R^2T^4$. The cooling rate is then $-\frac{dT}{dt} = -\frac{1}{R^3}\frac{dU}{dt}$ or

Answer: B

3. Let $T(t)$ be the temperature of the planet after time t and let T_0 be the temperature at time $t = 0$. Calculate the dependence of T/T_0 on t assuming $T_{\text{env}} = 0$ is the temperature of the planet's environment. Below, α is a constant.

   (a) $(1 + \alpha t)^{-1/3}$

   (b) $(1 + \alpha t)^{-1/2}$

   (c) $1 - \alpha t$

   (d) $(1 - \alpha t)^2$

   (e) $1 - e^{-\alpha t}$

Solution: We have $dT/dt \sim -T^4/R$, which integrates to give

Answer: A

4. Suppose a rocket around a star of mass M wishes to execute an orbital transfer from a circular orbit with radius R to a larger one with that of radius $8R$. One common way to do this is known as a *Hohmann transfer*, which has an intermediate elliptical orbit. This process requires two burns, with total delta-v $\Delta v = k\sqrt{\frac{GM}{R}}$ for some k. Assuming instantaneous burns, compute k.

   (a) $1 - \frac{1}{2\sqrt{2}}$

   (b) $\frac{1}{2} - \frac{1}{2\sqrt{2}}$

   (c) $\frac{1}{6} + \frac{1}{2\sqrt{2}}$

   (d) $\frac{3}{\sqrt{10}} - \frac{1}{2}$

   (e) $\frac{3}{\sqrt{10}} - \frac{1}{4}$

Solution: Note: we shall solve this question more generally with αR and then plug in $\alpha = 8$.

The vis-viva equation tells us that

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Plugging in $r = a = r' = R$ and $a' = (\alpha+1)R$ for the first burn, we get

$$\Delta v_1 = \sqrt{\frac{GM}{R}}\left(\sqrt{\frac{2\alpha}{\alpha+1}} - 1\right)$$

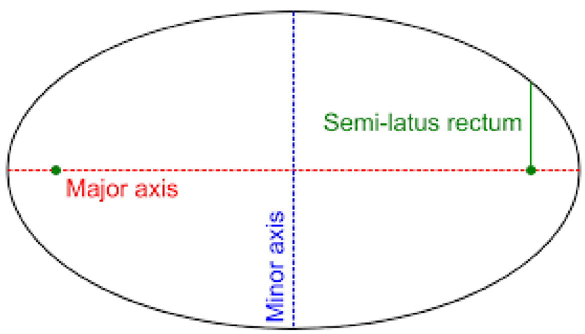
For the second burn we use $r = r' = a' = \alpha R$ and $a = (\alpha+1)R$ to obtain

$$\Delta v_2 = \sqrt{\frac{GM}{R}}\left(\frac{1}{\sqrt{\alpha}}\left(1 - \sqrt{\frac{2}{\alpha+1}}\right)\right)$$

Adding these together gives us $\Delta v = \sqrt{\frac{GM}{R}}\left(\sqrt{\frac{2\alpha}{\alpha+1}} - 1 + \frac{1}{\sqrt{\alpha}}\left(1 - \sqrt{\frac{2}{\alpha+1}}\right)\right)$ Plugging in $\alpha = 8$ gives us $\Delta v = \frac{1}{6} + \frac{1}{2\sqrt{2}}$.

Answer: C

5. It may be useful to know the *semi-latus rectum* of an ellipse is the distance between one of its foci and the point on the ellipse immediately above or below it, as shown in the diagram below.



Consider a highly eccentric planet with a semi-latus rectum that is nearly a hundred times smaller than its semi-major axis. What is its eccentricity?

(a) 0.99
(b) 0.995
(c) 0.9999
(d) 0.99995
(e) 0.999999

Solution: By constructing a right triangle, using the two foci and the point on the semi-latus rectum, we can find that the semi-latus rectum length ℓ is related to the semi-major axis a and the eccentricity e by

$$y^2 = \ell^2 = 4a^2e^2$$

where y is the distance between the point on the semi-latus rectum and the other focus (the one not directly below it), i.e, the hypotenuse of the triangle. Since an ellipse is defined such that all points on its circumference have a constant sum of distances from the foci $2a$, we also have that

$$y + \ell = 2a$$

We can now solve these equations, most easily by using difference of squares on the first and then dividing by the second to obtain $y - \ell = 2ae^2$, and it is now easy to see that $\ell = a(1-e^2)$. We now rearrange for $e = \sqrt{1 - \frac{\ell}{a}} \approx 1 - \frac{1}{2}\frac{\ell}{a}$. Plugging in $\frac{\ell}{a} = 0.01$, we get 0.995.

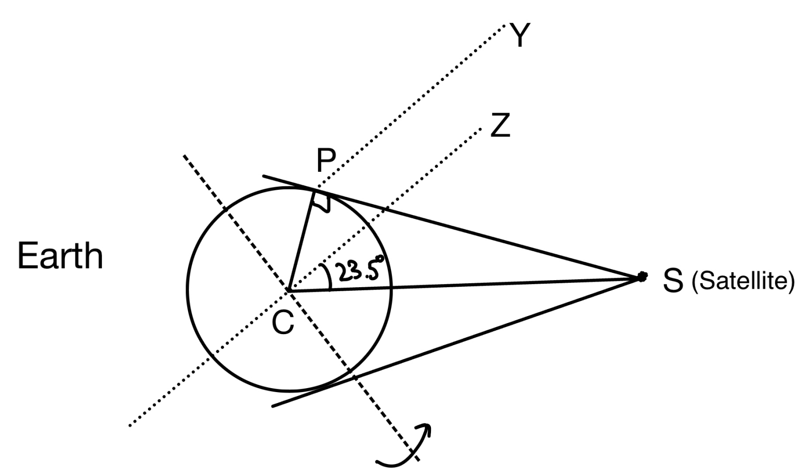
Answer: B

6. Galileo the Tyrant has conquered the Earth. In his madness, he launched many bright satellites into space because he thought that the celestial equator should be visible from Earth, and not just be an imaginary line. These satellites all orbit around Earth $3R_E$ from the surface, where R_E is the radius of earth.

   However, there was a major mistake made in the plan – the satellites now orbit in a circle on Earth's orbital plane instead of directly above the equator. Nevertheless, Galileo still forces all astronomers to use these satellites as the new Celestial Equator. They must calculate the declination for any star by finding the angle to the satellite at the same Right Ascension. What is the maximum absolute difference between the new declination and the old declination of bodies throughout the sky experienced by astronomers around the Earth? (Choose the closest value.)

   (a) 38°
   (b) 0°
   (c) 23.5°
   (d) 47°
   (e) 14.5°

Solution:



The difference in new and old declination for any body is the same as the angle between the old (or actual) celestial equator and the band of satellites for any location on Earth. However, the location of the satellites in the celestial sphere will change depending on the location on Earth because they are only a finite distance away, i.e. parallax.

Any location on the Celestial Equator is seen to be parallel to line CZ in the figure (since the projection of the night sky is considered to be at 'infinity' not a mere $4R_E$ away) from anywhere on Earth. That is, if an astronomer at any place on Earth is looking at the Celestial Equator, their sight will be parallel to CZ, e.g. PY.

However, the direction of the satellites in the sky that different parts of the Earth perceive will be different. Some examples of these lines of sight are PS and CS in the figure. The minimum and maximum values can be found using tangents to the Earth as shown. Notably the orbital plane, which the satellites are on, is 23.5 degrees off from the celestial equator (the angle between CS and CZ is 23.5°).

The maximum angle that we want is $\angle YPS = \angle ZCS + \angle PSC$.

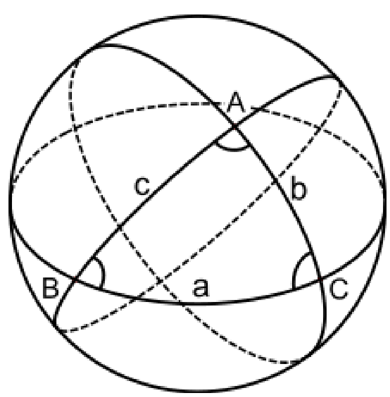
Here, $\angle PSC$ is the maximum angle for the satellites with reference to the orbital plane and is $\angle PSC = \arcsin CP/CS$. Note CS is measured from the center of Earth, and is equal to $4R_E$. So, this value is around 14.5 degrees. Thus the max difference is 23.5 + 14.5 = 38 degrees.

Answer: A

7. What is the geodesic distance between Boston (42°21′37" N, 71°3′28" W) and Mumbai (19°04′34" N, 72°52′39" E)? *Assume that the Earth is perfectly spherical.*

   (a) 2250 km
   (b) 6250 km
   (c) 10250 km
   (d) 12250 km
   (e) 15250 km

Solution: Look at the picture of a spherical triangle below:



If we project it to Earth, point A would be the North Pole, point B would be Boston, and point C would be Mumbai. From this, we can infer that side a is the geodesic distance. With the cosine formula:

$$\begin{aligned}
\cos a &= \cos b \cos c + \sin b \sin c \cos A \\
&= \cos(90 - \phi_{\text{Boston}}) \cos(90 - \phi_{\text{Mumbai}}) \\
&\quad + \sin(90 - \phi_{\text{Boston}}) \sin(90 - \phi_{\text{Mumbai}}) \cos(\lambda_{\text{Boston}} - \lambda_{\text{Mumbai}}) \\
a &\approx 110^\circ 8' 20" \approx 1.9223 \text{ rad}
\end{aligned}$$

Now, convert the geodesic distance from angle to length unit

$$\begin{aligned}
\text{distance} &\approx a \times R_\oplus \\
&\approx 1.9223 \times 6371 \text{ km} \\
&\approx 12250 \text{ km}
\end{aligned}$$

Answer: D

8. Which of the following statements are correct regarding spectral broadening?

- **P:** In the limit of zero temperature, spectral lines have infinitesimally narrow linewidth.
- **Q:** Consider two hypothetical spectral lines: line I has central frequency f_0 and full width at half maximum (FWHM) $w \ll f_0$, with only homogeneous broadening; line II has identical central frequency and FWHM but only has inhomogeneous broadening. Let $P_{\text{I,II}}(f)\Delta f$ denote the probability of measuring frequency f for each line within some bandwidth $\Delta f \ll w$. Then, $P_{\text{I}}(2f_0)\Delta f < P_{\text{II}}(2f_0)\Delta f$.
- **R:** The same spectral line of two ideal gases, one with temperature T and the other with temperature $2T$, are probed. The gases are otherwise identical. The linewidth of the hotter gas under thermal Doppler broadening is twice that of the cooler gas.

(a) None
(b) P only
(c) Q only
(d) Both P and Q
(e) Both Q and R

Solution: All spectral lines will have natural linewidth at zero temperature due to spontaneous emission, so statement P is incorrect.

Homogeneous broadening follows a Lorentzian lineshape, while inhomogeneous broadening follows a Gaussian lineshape. The latter has fatter tails and therefore larger probability density for outliers, so statement Q is incorrect.

Finally, the linewidth for thermal Doppler broadening goes as $\sqrt{T}$, so statement R is also incorrect.

Answer: A

9. Which of the following statements are true about a cluster of main sequence stars:

   - **A:** For two star clusters of the same total mass, the one with the larger number of stars will be brighter as measured by luminosity per unit mass.
   - **B:** The main-sequence turnoff point for star clusters is at higher temperatures if they are older.
   - **C:** The main-sequence turnoff point for star clusters is at lower luminosities if they are older.

   Assume the mass-luminosity relationship $L \propto M^{3.5}$ for main-sequence stars.

   (a) A and C
   (b) A only
   (c) C only
   (d) A, B, and C
   (e) A and B

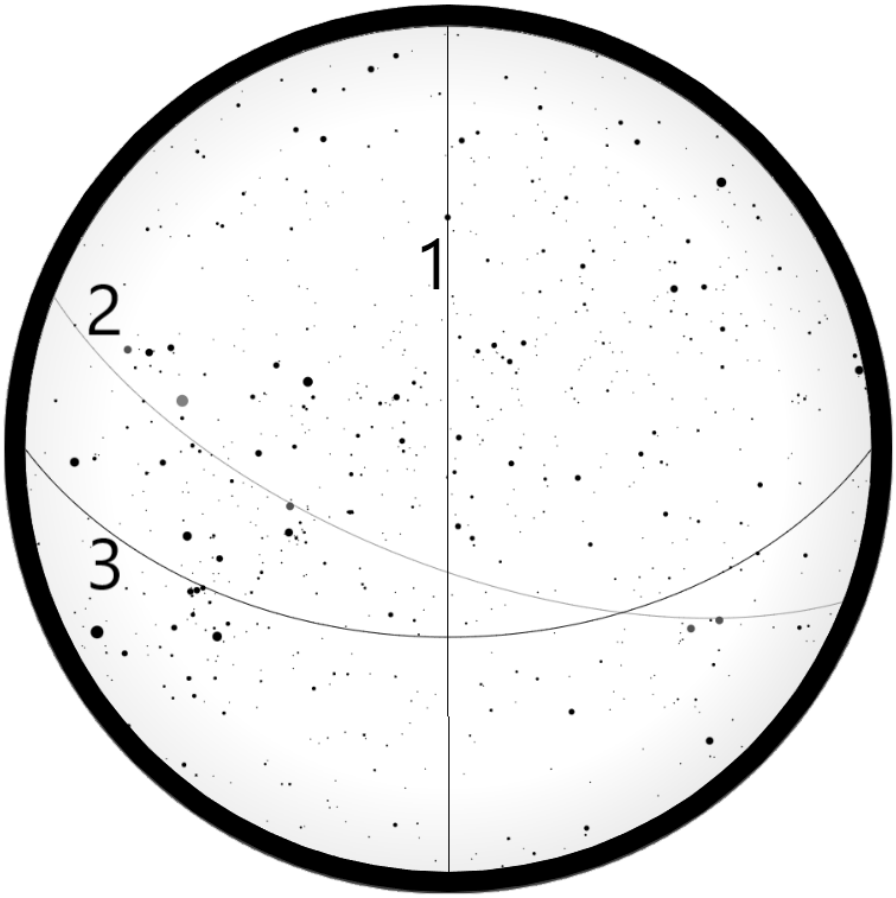
   Solution: A is false. The luminosity per unit mass L/M is proportional to $M^{2.5}$ with our given relationship (generally, $L/M \propto M^{\alpha}$ where $2.5 < \alpha < 3$). Thus this value will be larger when there are fewer large stars, as opposes to many smaller stars.

   B is false. The turnoff point for star clusters is at higher temperatures if they are younger. High temperature main sequence stars are usually short lived stars and thus will not exist in an old star cluster.

   C is true. Low luminosity stars are usually long lived main sequence stars and thus will still exist in an old star cluster.

   Answer: C

10. Match each imaginary line in the sky map below to its label.



(a) 1 - Ecliptic; 2 - Celestial Equator; 3 - Local Meridian

(b) 1 - Local Meridian; 2 - Ecliptic; 3 - Galactic Equator

(c) 1 - Galactic Equator; 2 - Celestial Equator; 3 - Ecliptic

(d) 1 - Local Meridian; 2 - Ecliptic; 3 - Celestial Equator

(e) 1 - Local Meridian; 2 - Celestial Equator; 3 - Ecliptic

Solution:

Line 1 connects the North and the South cardinal directions and goes through the center of the map (the zenith), so it must correspond to the local meridian. A good reference to use to observe that this line connects the North and South directions is that the star Polaris (α UMi) is extremely close to to this line.

Line 2 goes through zodiac constellations such as Cancer, Gemini, Aries, Pisces, and Aquarius. Therefore, this line is the Ecliptic. Another way of identifying that this line is the Ecliptic is to notice that there are several planets close to it.

Line 3 connects the East and West cardinal directions. Moreover, another good reference is that this line is very close to the Orion belt, more specifically to the star Mintaka (δ Ori). Therefore, this is the Celestial Equator.

Answer: D

11. **The following two questions build on each other**. An astronomer was studying the exoplanets orbiting a star with a mass of $10M_\odot$. The astronomer decided to draw a $\log T\, vs.\, \log a$ plot for the exoplanet orbits, where T corresponds to the period in years and a corresponds to the semi-major axis in AU. What would be the slope of the best fit line to this plot?

    Note that log represents the base 10 logarithm.

    (a) 4/3
    (b) 3/2
    (c) 1/10
    (d) 1
    (e) 1/2

Solution:

Using the units of years, AU, and $M_\odot$ for the period, semi-major axis, and mass, respectively, it is possible to write Kepler's 3^{rd} Law as follows:

$$\frac{T^2}{a^3} = \frac{1}{M}$$

rearranging the equation and taking the logarithm of both sides:

$$T^2 = \frac{a^3}{M}$$

$$\log T^2 = \log\left(\frac{a^3}{M}\right)$$

$$2\log T = 3\log a - \log M$$

$$\log T = \frac{3}{2}\log a - \frac{1}{2}\log M$$

$$\log T = \frac{3}{2}\log a - \frac{1}{2}\log 10$$

$$\log T = \frac{3}{2}\log a - \frac{1}{2}$$

The expression above shows that the slope of the best fit line is 3/2.

Answer: B

12. What would be the y-intercept of the best fit line to this plot?
   (a) 1
   (b) -4/3
   (c) -1/2
   (d) 0
   (e) -2/3

Solution:

The expression found on the previous question shows that the y-intercept of the best fit line is -1/2.

Answer: C

13. A recently observed exosolar system consists of a star, a planet, and the planet's satellite. The satellite has a revolution period of 100 minutes around the planet, and the planet has a 90 day revolution period around the star. The satellite approaches the surface of the planet to a minimum height of 1000 km and recedes to a maximum height of 7000 km. The radius of the planet is 3000 km. If the ratio of the mass of the star to the mass of the planet is 1×10^5, what is the semi-major axis of the planet's revolution around the star? Assume that the mass of the satellite is much smaller than the mass of the planet.
   (a) 1.86×10^7 km
   (b) 2.36×10^7 km
   (c) 2.86×10^7 km
   (d) 3.36×10^7 km
   (e) 3.86×10^7 km

Solution: The semi-major axis of the satellite's orbit around the planet is $a_{satellite} = (7000 + 1000 + 3000 \times 2)/2 = 7000$ km. If the masses of the star, planet and satellite are M, m, and m_{sat}, Kepler's 3rd law yields:

$$100\,\text{minutes} = 2\pi\sqrt{\frac{(a_{satellite})^3}{G(m+m_{sat})}} \tag{1}$$

$$90\,\text{days} = 2\pi\sqrt{\frac{(a_{planet})^3}{G(M+m)}} \tag{2}$$

Dividing equation 2 by 1,

$$\left(\frac{90 \times 60 \times 24}{100}\right)^2 = \frac{a^3_{planet}}{a^3_{satellite}}\left(\frac{m+m_{sat}}{M+m}\right) \approx \frac{a^3_{planet}}{(7000\text{ km})^3}\left(\frac{1}{1 \times 10^5 + 1}\right) \tag{3}$$

Note: $m_{sat} << m << M$

Answer: E

14. **The following three questions build on each other.** A cylindrical space station rotating about its primary axis could create artificial "spin gravity," allowing people to live on the inner surface of the cylinder. For such a station with a radius of 50 m, what rotational period is needed for inhabitants to experience spin gravity equal to Earth's gravity $g = 9.81\,\text{m/s}^2$?

    (a) 2.3 s
    (b) 5.1 s
    (c) 7.1 s
    (d) 14.2 s
    (e) 32.0 s

Solution: The centrifugal acceleration is given by $a = \omega^2 r$, where ω is the angular velocity. If $a = g$ and $r = 50\,\text{m}$, then $\omega = 0.44\,\text{s}^{-1}$.

To convert the angular velocity into a period, we use $T = \frac{2\pi}{\omega} = 14.2\,\text{s}$.

Answer: D

15. If the space station is too small, inhabitants might experience unwanted effects of being in a rotating reference frame, such as the Coriolis "force" on moving objects: $\vec{F} = -2m\vec{\omega} \times \vec{v}$, where $\vec{\omega}$ is the angular velocity vector of the space station and $\vec{v}$ is the velocity of the object in the rotating reference frame.

    For a space station with spin gravity equal to Earth's gravity, what is the minimum radius needed so that a person walking at 1 m/s in any direction experiences negligible Coriolis force ($< 1\%$ of Earth's gravity)?

    (a) 10 m
    (b) 20 m
    (c) 41 m
    (d) 1.0 km
    (e) 4.1 km

Solution: Consider the "worst case" scenario, where the person is walking perpendicularly to the angular velocity vector. The magnitude of the Coriolis force is then $F_c = 2m\omega v$.

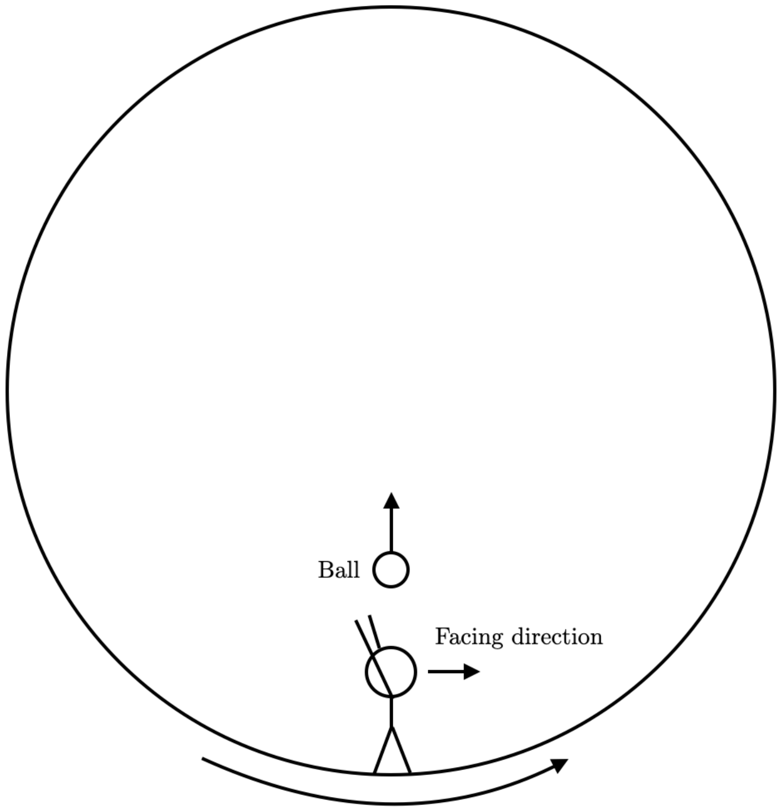
Given that the station is generating 1 G of gravity, the magnitude of the gravitational force is just $F_g = mg$. Also, just as in the previous question, we can write $\omega = \sqrt{\frac{g}{r}}$.

Then the ratio of the two forces is

$$\frac{F_c}{F_g} = \frac{2m\omega v}{mg} = \frac{2v}{\sqrt{gr}}$$

We want $\frac{F_c}{F_g} < 0.01$. Plugging in $g = 9.81\,\text{m/s}^2$ and $v = 1\,\text{m/s}$ gives $r > 4.1\,\text{km}$.

Answer: E



16. You are standing inside the space station, facing in the direction of the cylinder's rotation (see image). You throw a ball directly upward. Where does it land?

    (a) Right back in your hand
    (b) In front of you
    (c) Behind you
    (d) To your right
    (e) To your left

Solution: This is just an exercise in carefully using the right-hand rule. It's helpful to draw a picture.

If you are facing in the direction of the spin, the angular velocity vector points towards your right. For clarity, let's define a local right-handed coordinate system: the angular velocity vector is in the $\hat{x}$ direction, you are facing in the $\hat{y}$ direction, and the $\hat{z}$ direction is up (towards the center of the cylinder.) Note that we can immediately rule out (d) and (e) due to symmetry considerations: the motion must be confined to the yz plane.

When the ball is moving upwards ($+\hat{z}$), the Coriolis force is in the direction $-\hat{x} \times \hat{z} = \hat{y}$. Similarly, when the ball is moving back downwards ($-\hat{z}$), this force is in the $-\hat{y}$ direction.

The net effect is that the ball gains some velocity component in the $+\hat{y}$ direction on the way up, and loses it on the way down. This means its landing position is deflected in the $+\hat{y}$ direction, or in front of you.

This problem can also be considered in an inertial reference frame, where you and the station are rotating. The ball's path from the throwing point to the eventual landing point must follow a chord of the circle (a straight line). However, your own path to the landing point is a circular arc, which is longer. Since you threw the ball in a direction perpendicular to your own velocity vector, its speed must also be greater than yours. Since the ball moves from the throwing point to the landing point along a shorter path *and* at a higher speed, it certainly gets there before you do. This means it lands in front of you.

Answer: B

17. Consider a giant star with radius R. The core has temperature T_c and the surface has temperature $T_s \ll T_c$. Given R, T_c, and T_s, Alice and Bob estimate the core's radius in two different ways, both assuming that the star is in thermal equilibrium:

- **Alice:** The core radiates energy.
- **Bob:** The inside of the star conducts heat with constant uniform thermal conductivity κ such that $\kappa \gg \frac{R\sigma T_s^4}{T_c}$.

Let R_A and R_B be the core radii estimated by Alice and Bob. Which of the following is the ratio R_A/R_B proportional to?

(a) T_s/T_c
(b) T_s^2/T_c^2
(c) $1/RT_cT_s^2$
(d) $R^2/T_s^4T_c^2$
(e) $T_s^4T_c^2/R$

Solution: For both observers, the star emits power $P = 4\pi R^2\sigma T_s^4$.

For Alice, note that the core radiates with power $4\pi R_c^2 T_c^4$. Since the star is in thermal equilibrium, this equates with P, giving $R_c = \frac{T_s^2}{T_c^2}R$.

For Bob, at radius r from the center of the star, we have the heat equation $P = -\kappa(4\pi r^2)\frac{dT}{dr}$. Thus, we see that

$$\int_{R_c}^{R} \frac{P dr}{r^2} = -\int_{T_c}^{T_s} 4\pi\kappa dT$$

$$\implies P\left(\frac{1}{R} - \frac{1}{R_c}\right) = 4\pi\kappa(T_s - T_c) \approx -4\pi\kappa T_c$$

$$\implies R_c = \frac{1}{\frac{4\pi\kappa T_c}{P} + \frac{1}{R}} \approx \frac{P}{4\pi\kappa T_c} = \frac{R^2\sigma T_s^4}{\kappa T_c}.$$

Therefore, the ratio in the two radii is

$$\frac{R_A}{R_B} = \frac{\alpha}{RT_s^2T_c}.$$

Answer: C

18. Connor is stranded on an island and wishes to determine his latitude, but he only wakes up at sunrise and sunset. He constructs a vertical stick in the ground. On some day, he measures the angle between the shadows made at sunrise and sunset as 105°. Which of the following intervals of latitudes could Connor be in?

    (a) $(0^\circ\text{ N}, 10^\circ\text{ N})$
    (b) $(10^\circ\text{ N}, 20^\circ\text{ N})$
    (c) $(20^\circ\text{ N}, 30^\circ\text{ N})$
    (d) $(30^\circ\text{ N}, 40^\circ\text{ N})$
    (e) $(40^\circ\text{ N}, 50^\circ\text{ N})$

Solution: We will derive the expression for the change in angle of the stick's shadow. Note that it is equivalent to the change in azimuth of the Sun from sunrise to sunset.

Let the Sun (denoted as S) have declination δ on some day. Denote the North Celestial Pole as N and the zenith as Z. Then, $\widehat{NZ} = 90^\circ - \phi$, where ϕ is the latitude. Also, $\widehat{NA} = 90^\circ - \delta$ and $\widehat{ZA} = 90^\circ$ at sunrise and sunset. Using the spherical triangle NAZ, we see that

$$\cos\left(\widehat{NA}\right) = \cos\left(\widehat{ZN}\right)\cos\left(\widehat{ZA}\right) + \sin\left(\widehat{NZ}\right)\sin\left(\widehat{NA}\right)\cos\left(\angle NZA\right)$$

giving

$$\cos\left(\angle NZA\right) = \frac{\sin\delta}{\cos\phi}.$$

Then, we see that the change in angle is either

$$2\arccos\left(\frac{\sin\delta}{\cos\phi}\right) \text{ or } 2\pi - 2\arccos\left(\frac{\sin\delta}{\cos\phi}\right),$$

where δ is the declination of the Sun and ϕ is the latitude. Since $105^\circ < 180^\circ$, we only need to focus on the first expression.

From this, we can see that, as ϕ is increased, the range of values taken by $2\arccos\left(\frac{\sin\delta}{\cos\phi}\right)$ keeps increasing, noting that the Sun has maximum declination 23.44°. It is always upper bounded by 180° and its lower bound decreases as ϕ increases. Since there is a unique answer to this question, that means it is possible Connor's latitude is in the interval $(40^\circ\text{ N}, 50^\circ\text{ N})$.

We can check that, at 40° N, the lower bound is 117.4°, when the Sun has maximum declination 23.5°. Also, at 50° N, the lower bound is 103.5°.

Answer: E

19. Which of the following statements CANNOT be inferred from Kepler's laws of motion?

    - **I:** A planet moves in an elliptical orbit around the Sun.
    - **II:** The eccentricities of the orbits of all solar system planets are small.

- **III:** A solar system planet has its highest tangential velocity when it is closest to the Sun.
- **IV:** All planets move in elliptical orbits in roughly the same plane around the Sun.

(a) I only
(b) IV only
(c) II, III, and IV
(d) II and IV
(e) II and III

Solution: I is Kepler's First Law. III is a corollary of Kepler's Second Law. Kepler's Laws do not make any statement about the values of the eccentricities and the orientation of the orbital planes.

Answer: D

20. Just before dawn of the summer solstice, Christopher the sailor begins sailing due West from $50^\circ N, 5^\circ W$. His watch is set to UTC. On seeing his 91st sunrise at sea, his watch reads 3:00am. What is his longitude? (The equation of time, in the convention solar time minus mean time, is -3 minutes on the summer solstice and 8 minutes on the autumnal equinox)

(a) $43^\circ W$
(b) $45^\circ W$
(c) $48^\circ 15' W$
(d) $50^\circ W$
(e) $53^\circ 15' W$

Solution: The idea presented in this problem is that timekeeping systems can be used to navigate and measure longitude.
The question asks for Christopher's longitude on the autumnal equinox. The equation of time gives:

$$\mathrm{LST} - \mathrm{LMT} = 8^m$$

where LST is local solar time and LMT is local mean time. LMT is related to GMT, which is also UTC, by longitude (λ).

$$\mathrm{LMT} - \mathrm{GMT} = \lambda$$

Thus,

$$\mathrm{LST} = \mathrm{GMT} + \lambda + 8^m$$

At sunrise on an equinox, LST $= 6^h$. Thus, $\lambda = 43^\circ W$.

Answer: A

21. What is the largest range of latitudes ϕ for which, at some time during the day, the line between Rigel ($\alpha_R = 5^h 14^m 32^s,\ \delta_R = -8^\circ 12' 5.9"$) and Betelgeuse ($\alpha_P = 5^h 56^m 33^s,\ \delta_B = 7^\circ 24' 40.3"$) appears vertical? Answer to within 5 degree accuracy.

(a) $-15^\circ < \phi < 15^\circ$

(b) $-25^\circ < \phi < 25^\circ$

(c) $-35^\circ < \phi < 35^\circ$

(d) $-45^\circ < \phi < 45^\circ$

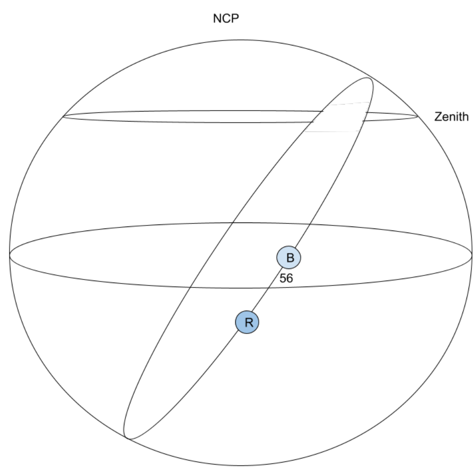
(e) $-55^\circ < \phi < 55^\circ$

Solution: The key to this problem is that Rigel and Betelgeuse define a great circle, and they appear vertical when the Zenith is on this great circle.
Rigel and Betelgeuse are very close to each other in the sky, so the region they occupy on the celestial sphere can be approximated to be flat. The slope that they make relative to the equator is approximately

$$\arctan \frac{\delta_B - \delta_R}{\alpha_B - \alpha_R} = 56^\circ.$$

The angle that their great circle makes with the celestial equator is this value. The Right Ascension of the intersection turns out not to matter.
Over the course of a day, the Zenith travels along the small circle a height ϕ above the equator, where ϕ is the latitude. The maximum latitude for which this is possible is 56°, as seen in this figure.

$\boxed{\textbf{Answer: E}}$



22. Observed from the equator, what is the maximum duration over which Venus can be observed (continuously) after sunset? Assume that the orbits of Venus and Earth are circular of radius 0.723 AU and 1 AU and lie on the ecliptic plane.

(a) 2^h50^m

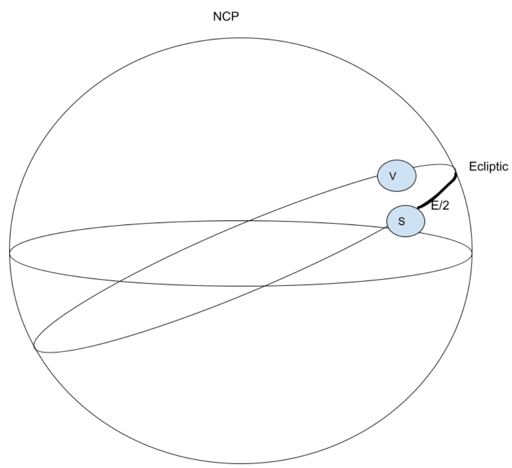
(b) 3^h5^m

(c) 3^h20^m

(d) 3^h40^m

(e) 12^h

Solution: From the equator, the difference in set times between celestial bodies is just their difference in Right Ascension. We thus wish to find a solar system configuration that maximizes $\Delta\alpha = \alpha_{Sun} - \alpha_{Venus}$.
We do not provide a formal mathematical proof here, neither did we expect test-takers to rigorously arrive at their answer. Rather, we hoped test-takers would use their intuition to find this configuration.
Given enough time, the Sun and Venus can take any pair of locations on the Ecliptic, with the restriction that the separation, or elongation, does not exceed $E = \arcsin \frac{a_{Venus}}{a_{Sun}} = 46.3°$. The configuration is the following:



It should be clear that Venus and the Sun should be at maximum separation E. Furthermore, in this figure, the arc between Venus and the Sun is essentially parallel to the equator and also high as possible on the sphere. We thus would expect an answer close to $\frac{E}{cos\varepsilon}$, and indeed, some spherical geometry gives:

$$2 \arcsin \frac{\sin E/2}{\cos\varepsilon\sqrt{1 + \tan^2\varepsilon \sin^2 \frac{E}{2}}}$$

This evaluates to $3^h 20^m$.

$\boxed{\textbf{Answer: C}}$

23. The Earth orbits the Sun with orbital eccentricity $e = 0.0167$. What percent more total solar irradiance does the Earth receive at perihelion compared to aphelion?

(a) 0.028%
(b) 0.056%
(c) 1.7%
(d) 3.4%
(e) 6.9%

Solution: The distance from the Sun at aphelion is $a(1+e)$, and at perihelion is $a(1-e)$. The solar irradiance the Earth receives is proportional to the inverse square of its distance from the Sun. The semi-major axis a cancels to yield a ratio of $\left(\frac{1+e}{1-e}\right)^2$. Plugging in e for the Earth, solar irradiance is about 7% higher at perihelion compared to aphelion. (Since perihelion is in January, this has the effect of making seasonal extremes slightly stronger in the Southern Hemisphere than the Northern Hemisphere.)

Answer: E

24. Consider a main sequence star 500 times less luminous than the sun. This star must have a ______ core and a ______ envelope.

   (a) convective, convective
   (b) convective, radiative
   (c) radiative, convective
   (d) radiative, radiative
   (e) static, static

Solution: There is, roughly speaking, a cubic dependence of luminosity on mass. Thus, a star 500 times less luminous than the sun is roughly 8 times less massive. Stars this small (below roughly a third to a quarter of a solar mass) are known to be purely convective; this is largely due to their opacity to radiation.

Answer: A

25. Evan uses a telescope of focal distance $f = 1000\,\text{mm}$ to observe a star known to have a parallax $p = 50\,\text{mas}$. Observing the CCD of the telescope, Evan notices that the star has a diameter $D = 0.1\,\mu m$. Furthermore, he measures its apparent magnitude to be $m = 1$. What is the approximate temperature of this star? Please, neglect the atmospheric seeing in your calculations.

   (a) 3000 K
   (b) 3500 K
   (c) 4000 K
   (d) 4500 K
   (e) 5000 K

Solution: The distance of the star in parsecs is

$$d = \frac{1}{p('')} = \frac{1}{50 \times 10^{-3}} = 20\,\text{pc}$$

Converting to meters:

$$d = 3.086 \times 10^{16} \times 20 = 6.172 \times 10^{17}\,\text{m}$$

The angular radius of the star in radians is

$$\theta = \frac{1}{f} \cdot \frac{D}{2} = \frac{1}{1} \cdot \frac{0.1 \times 10^{-6}}{2} = 5 \times 10^{-8}$$

Converting to meters:

$$R = \theta d = 6.172 \times 10^{17} \times 5 \times 10^{-8} = 3.086 \times 10^{10}\,\text{m}$$

Calculating the apparent magnitude of the Sun:

$$m_\odot - M_\odot = 5 \log a_\oplus - 5$$

$$m_\odot = 4.83 + 5 \log \frac{1}{206265} - 5 = -26.74$$

Comparing the star to the Sun using Pogson's equation:

$$m - m_\odot = -2.5 \log \left(\frac{F}{F_\odot} \right)$$

$$F = F_\odot \cdot 10^{0.4(m_\odot - m)}$$

$$\frac{4\pi R^2 \sigma T^4}{4\pi d^2} = \frac{L_\odot}{4\pi a_\oplus^2} \cdot 10^{0.4(m_\odot - m)}$$

$$T = \left(\frac{d^2 L_\odot}{4\pi\sigma R^2 a_\oplus^2} \cdot 10^{0.4(m_\odot - m)} \right)^{1/4}$$

$$T = \left(\frac{(6.172 \times 10^{17})^2 \times 3.85 \times 10^{26}}{4\pi \times 5.671 \times 10^{-8} \times (3.086 \times 10^{10})^2 \times (1.496 \times 10^{11})^2} \cdot 10^{0.4 \times (-26.74 - 1)} \right)^{1/4}$$

$$\boxed{T \approx 3000\,\text{K}}$$

Answer: A

26. Two exoplanets, A and B, have circular orbits around the same central star. Suppose that the ascending nodes of the orbits are located at the same ecliptic longitude, defined analogously as the one for the solar system, and that both planets are at that point in the beginning. How long will it take for the planets to have an equal, common ecliptic longitude again, knowing that the inclinations of their orbits are $i_1 = 30^\circ$ and $i_2 = 70^\circ$, and that their periods are $T_1 = 2\,\text{yr}$ and $T_2 = 1\,\text{yr}$?

(a) 42 days
(b) 44 days
(c) 46 days
(d) 48 days
(e) 50 days

Solution: From the following spherical triangle, letting λ be the ecliptic longitude of a planet and ω its angular velocity along its orbit:

$$\cot 90^\circ \sin i + \cos i \cos \lambda = \cot \omega t \sin \lambda$$

$$\tan \lambda = \cos i \tan \omega t$$

Hence, when the ecliptic longitudes are the same again:

$$\cos i_1 \tan \omega_1 t = \cos i_2 \tan \omega_2 t$$

Since $\omega_2 = 2\omega_1$:

$$\cos i_1 \tan \omega_1 t = \cos i_2 \tan 2\omega_1 t$$

$$\cos i_1 \tan \omega_1 t = \cos i_2 \frac{2\tan \omega_1 t}{1 - \tan^2 \omega_1 t}$$

$$\tan \omega_1 t = \sqrt{1 - \frac{2\cos i_2}{\cos i_1}}$$

$$t = \frac{1}{\omega_1} \arctan \sqrt{1 - \frac{2\cos i_2}{\cos i_1}}$$

$$t = \frac{T_1}{2\pi} \arctan \sqrt{1 - \frac{2\cos i_2}{\cos i_1}}$$

$$\boxed{t \approx 0.137\,\text{yr} \approx 50\,\text{days}}$$

Answer: E

27. Cosmologists consider three possible universes: universes dominated by *baryonic matter* (i.e. ordinary matter), universes dominated by radiation, and universes dominated by dark energy (realistically, we consider some mix of the above). An equation written by Alexander Friedmann implies that for a matter-dominated universe, the Hubble parameter is proportional to the square root of that universe's matter density. In a matter-dominated universe, how does the size of the universe evolve with time t?

(a) constant
(b) $\sqrt{t}$
(c) $t^{2/3}$
(d) t
(e) e^t

Solution: The Hubble parameter relates proper distance to recessional velocity; equivalently, one can see it as relating the size of the universe R to the change in that size $\dot{R}$. Another thing to notice is that since the amount of matter in the universe doesn't change, its density is $\rho \propto R^{-3}$. Thus we can write Friedmann's equation as

$$\frac{\dot{R}}{R} = R^{-3/2}$$

Notice we're being liberal with constants here, because we're only interested in the asymptotic behavior of $R(t)$. Thus $\frac{1}{R}\frac{dR}{dt} = \frac{1}{R^{3/2}}$, so we rearrange to obtain

$$\int \sqrt{R} dR = \int dt$$

so that $R \propto t^{2/3}$.

Answer: C

28. Suppose leap days were removed from the calendar, so that each calendar year has exactly 365 days. In the first year of this calendar change, summer solstice is on June 21st. After 100 years of this calendar, what is the date of summer solstice?
    (a) May 28th
    (b) June 13th
    (c) June 21st
    (d) June 29th
    (e) July 15th

Solution: A tropical year is slightly longer than 365 solar days (0.2422 day longer, which is not on the constants table, but is not needed to solve the problem). Without leap days, the solstice will drift a bit later each year. In any 100 year period, there are typically 24 leap days because there is a leap day every 4 years, except for the century years that are not divisible by 400. 24 days after June 21st is July 15th.

Answer: E

29. It is observed that a planet orbiting a star increases the observed magnitude of the system by 0.005 when it transits across the star. Assuming that the system is viewed edge on and is orthogonal to the line of sight, determine the radius of the planet R_p in terms of the radius of the star R.
    (a) $0.056R$
    (b) $0.067R$
    (c) $0.089R$
    (d) $0.12R$
    (e) $0.15R$

Solution: The star presents an effective area of πR^2 for radiation when the planet does not obstruct any radiation. When the planet transits across the star, the effective area of radiation is $\pi R^2 - \pi R_p^2$ Change in magnitude Δm can then be expressed using the Pogson equation:

$$m_1 - m_2 = \Delta m = -2.5 \log_{10}\left(\frac{R^2 - R_p^2}{R^2}\right) \tag{4}$$

$$R_p = \sqrt{(1 - 10^{.005/-2.5})}\, R \approx 0.067R \tag{5}$$

Answer: B

30. What is the focal ratio of a telescope with 1.2 m in focal length and 12 cm in aperture?
    (a) f/1000
    (b) f/100
    (c) f/10
    (d) f/1
    (e) f/0.1

Solution: The focal ratio of a telescope can be calculated by this equation:

$$f = \frac{f}{D} \tag{6}$$

where f is the focal length of the telescope and D is the aperture/diameter. Hence,

$$f = 120/12 = 10$$

Answer: C