2022 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is 200.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on March 26th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

2 Short Questions

1. **(10 points)** The energies of an electron in a hydrogen atom are given by

$$E_n = -\frac{13.606 \text{ eV}}{n^2}$$

where $n = 1, 2, 3, \ldots$ represents the principal quantum number of the shell in which the electron is located.

The Ly-α spectral line is produced when an electron transitions from the $n = 2$ to the $n = 1$ energy level. Astronomers observe that the wavelength of the Ly-α line in a distant receding galaxy's emission spectrum is $\Delta\lambda = 7.13$ nm greater than the value measured in a lab.

Calculate the object's approximate distance from us in Mpc (assuming Hubble's constant $H_0 = 70$ km/s/Mpc).

2. **(10 points)** The following expression describes the mass function of a binary system:

$$f(M_1, M_2) = \frac{M_2^3 sin^3(i)}{(M_1 + M_2)^2}$$

- M_1: Mass of star 1.
- M_2: Mass of star 2.
- i: Inclination of the orbit.

Consider an **eclipsing** binary system with a period of 70 years and a total semi-major axis of 36 AU. In this system, the semi-major axis of star 1 is two times larger than the semi-major axis of star 2.

Estimate the mass function of the binary system in terms of solar masses.

3. **(10 points)** Consider a star A (apparent magnitude $m_A = 10.9$, radius $R_A = 0.42 R_\odot$). A periodic transiting event is observed to have a decrease the collected flux by 0.07 %. If this event was caused by a transiting exoplanet around star A, what would be the radius of that exoplanet in Earth radii?

4. **(10 points)** Posidonious from the first century BC estimated the circumference of the Earth by observing the rising and setting of the star Canopus. We will retrace his calculations in this problem. He observed Canopus on but never above the horizon at Rhodes. On the other hand, Canopus rose to a maximum of about 7.5° above the horizon at Alexandria. Assume Rhodes and Alexandria have the same longitude and the distance between the two cities is 800 km. Given only this information, estimate the radius of the Earth. How far off is it from the actual value of 6400 km. Justify your answer.

5. **(10 points)** There is an electron with its mass m_e that orbits a proton with mass m_p at a radius r. If we only assume the Coloumbic attraction,

(a) Write an expression of the total energy and the orbital momentum of the electron.

(b) Rewrite the expression of the total energy E in terms of the orbital momentum L, both from the part(a).

Use e for the electric charge quantity and assume that m_p is incomparably greater than m_e ($m_p \gg m_e$).

3 Medium Questions

1. **(30 points)** *The speed of light*

   The year is 1671 and you are astronomer Ole Rømer, measuring the period of Io's orbit around Jupiter by timing the passages of Io into or out of Jupiter's shadow.

   In December of 1671, Jupiter is at its first quadrature and you observe eclipses at the following times:

   - December 18 at 06:17:48
   - December 20 at 00:46:09
   - December 21 at 19:14:30

   In June of 1672, Jupiter is at its second quadrature and you observe eclipses at the following times:

   - June 19 at 08:42:50
   - June 21 at 03:11:30
   - June 22 at 21:40:10

   (a) **(2 points)** What is the interval between eclipses of Io as measured in December 1671, and what is it in June 1672?

   (b) **(7 points)** These orbital periods are slightly different. Rømer hypothesized that this is evidence that light has a finite speed. Explain why he thought this.

   (c) **(15 points)** Calculate the speed of light from these observations, and what you know about the orbits of Earth and Jupiter. Explain any simplifying assumptions that you make. How close is this speed of light to the correct value? (Hint: at second quadrature Earth is moving directly away from Jupiter, and at first quadrature Earth is moving directly towards Jupiter.)

   (d) **(6 points)** In 1672, Rømer did not have an accurate measurement of the distance from the Earth to the Sun. Write the speed of light *as he would have had to write it*, in terms of the unknown Earth-Sun distance a.

   (Be careful: Rømer also did not know the gravitational constant or the mass of the sun!)

2. **(30 points)** A meteorite that is radially approaching the Earth collides with a space station that revolves around the Earth in a circular orbit with radius R. For all parts of the question, express your results in terms of the mass M of the Earth, the gravitational constant G, the mass m_1 of the meteorite, and the mass m_2 of the space station.

   (a) Assume that, after the impact, the meteorite and the space station form a conglomerate that moves in a closed orbit which approaches the center of the Earth at a minimum distance $R/2$. State what the shape of the orbit of the conglomerate is and determine:

   (i) the speed of the meteorite just before the collision,

   (ii) the minimum and maximum speeds of the conglomerate,

   (iii) the maximum distance of the conglomerate from the center of the Earth.

   Determine the condition that m_1 and m_2 must obey so that the aforementioned scenario is possible.

   (b) Determine the minimum speed that the meteorite should have just before the collision so that the conglomerate moves in an open orbit after the impact. For this minimum value of the speed of the meteorite, state what the shape of the orbit of the conglomerate would be and determine

   (i) the maximum speed of the conglomerate,

   (ii) its minimum distance from the center of the Earth,

   (iii) the angle traversed by the orbital radius from the moment of the collision until the moment when the conglomerate approaches the center of the Earth to minimum distance.

4 Long Questions

1. **(45 points)** *The Curious Orbit of James Webb*

   For his upcoming Astrophysics Club presentation, Will researches the recently launched James Webb Space Telescope (JWST), the next-generation telescope designed as the successor of the Hubble Space Telescope. The largest space telescope ever built, the JWST uses its large collecting area to observe in the infrared spectrum. It orbits around the L_2 Lagrange point of the Earth-Sun system. Lagrange points are equilibrium points for a small body in the Earth-Sun system; L_2 is the point on the Earth-Sun line located beyond Earth's orbit.

   In the problem, let M and m be the mass of the Sun and Earth, respectively, with $M \gg m$. Additionally, consider the Sun's and Earth's radius to be $R_\odot$ and $R_\oplus$ respectively, and the Earth to orbit the Sun in a perfectly circular orbit of radius R.

   (a) **(2 points)** The orbit of JWST was designed to circle around L_2 in a big enough orbit to avoid Earth's shadow. What is the benefit of i) being at a Lagrangian point and ii) avoiding Earth's shadow?

   (b) **(5 points)** Taking first order approximations, about how far is L_2 from Earth? Express your answer both in terms of the variables defined and numerically, in km.

   (c) **(5 points)** In the rotating reference frame in which the Earth and the Sun are stationary, JWST orbits L_2 in the plane perpendicular to the Earth-Sun line that passes through L_2. If JWST orbits in a circle of radius r around L_2 in this frame, what is the minimum r that avoids the Earth's shadow at all times? Express your answer both in terms of the variables defined and numerically, in km.

   (d) **(20 points)** Consider a scenario where JWST is stationary in the aforementioned rotating reference frame and has a small displacement $\boldsymbol{\delta r} = \delta x \hat{\boldsymbol{\imath}} + \delta y \hat{\boldsymbol{\jmath}}$ relative to L_2, where $\hat{\boldsymbol{\imath}}$ is the unit vector along the Earth-Sun line away from the Sun and $\hat{\boldsymbol{\jmath}}$ is a unit vector perpendicular to $\hat{\boldsymbol{\imath}}$. Both $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are stationary in the rotating frame. To first order (i.e. assuming $|\boldsymbol{\delta r}| \ll x$), what is the acceleration of JWST in the rotating frame?

   (e) **(5 points)** The presence of the Coriolis force in the rotating reference frame destabilizes orbits around L_1, L_2, and L_3 while stabilizing orbits around L_4 and L_5. **Disregarding the Coriolis force for this part only,** are orbits stable around L_2 when there is no Coriolis force? Is this result generalizable? In other words, what can be said about the stability of orbits around an arbitrary, stationary point where there are no masses within the orbit and no fictitious forces involved?

   (f) **(8 points)** Suppose JWST orbits in the circle described in part (c) with a constant speed and an orbital radius of 500,000 km. Suppose further that the jet propulsion of the JWST is programmed to counteract only the Coriolis force; the rest of JWST's motion is due to the natural gravitational dynamics at L_2. Using the assumption that the first order expression derived in (d) still applies, if JWST has a mass of 6500 kg, what is the average magnitude of the force over a long period of time? The following averages (calculated from 0 to 2π) might be helpful:

   $$\overline{|\sin x|} = \frac{2}{\pi} \qquad \overline{\sin^2 x} = \frac{1}{2} \qquad \overline{|\sin^3 x|} = \frac{4}{3\pi}$$

   For reference, the magnitude of the Coriolis force is given as,

   $$|\boldsymbol{F}| = 2m|\boldsymbol{\omega} \times \boldsymbol{v}|$$

2. **(45 points)** *The Sundial I*

   While on a walk in Princeton University, Leo stumbled upon the following sundial, mounted on the southern wall of a building:

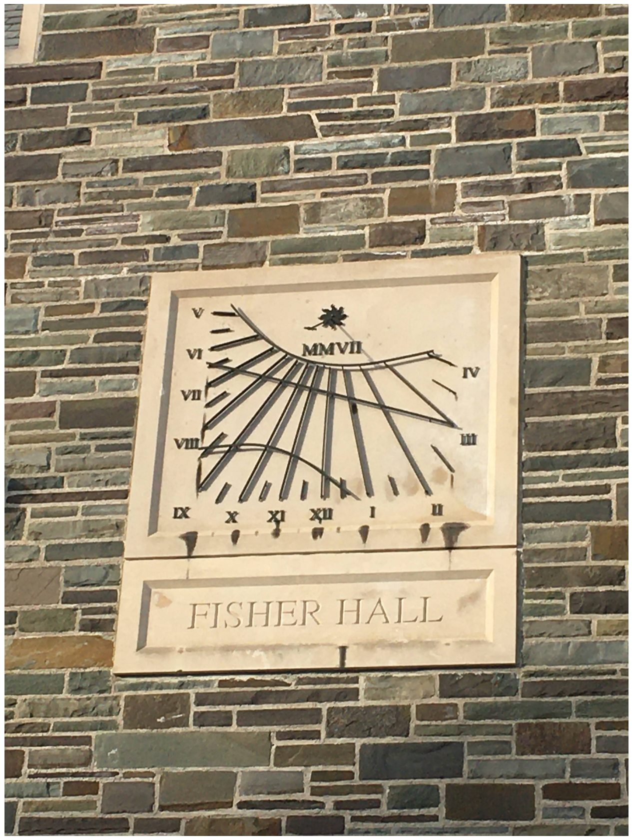

Figure 1: Picture taken by Leo Yao, December 2020.

He was familiar with the lines pointing outwards from the center, marking off time of day. However, he also noticed the three curves crossing the other lines. After a bit of thought, he realized these curves marked off the path of the shadow on the equinoxes and solstices.

(a) **(3 points)** For each of the two equinoxes and solstices, match the day to the curve (top, middle, bottom) denoting the path of the shadow on that day. (No explanation needed)

(b) **(1 point)** For the days corresponding to the middle curve, what is the declination of the Sun on those days? Assume the length of the day is small compared to the length of the year. (No explanation needed)

He then noticed that the middle curve seemed to be a straight line, and started thinking about if this is the case. He first considered a simpler system: a stick mounted vertically on a flat surface, casting a shadow on flat ground.

(c) **(9 points)** Consider the shadow of the tip of the stick, which might possibly trace a straight line over the course of the day. Explain why, if this happens, it can only happen on a day when the Sun's declination is that determined above.

This part can be solved independently, or as part of your solution for the next part. If your solution for the next part also proves this, note that down on your solution sheets, and proceed directly to the next part.

(d) **(24 points)** Prove that, for the Sun's declination determined above, the shadow of the tip of the stick traces a straight line over the course of the day.

Any method is acceptable, as long as it is presented clearly and rigorously. For example, one possible method might involve the following steps:

i) Determining the orientation of the line and explaining why it must be in this orientation;
ii) Determining the length of the shadow for a given position of the Sun in alt-az coordinates;
iii) Deriving a relation between altitude and azimuth given that the tip of the shadow is on the line;
iv) Determining a constant quantity and showing that it is constant over all positions of the Sun that day.

If you skipped the previous part, make sure your proof also shows the inverse: that for a different declination of the Sun, the tip's shadow does not trace a straight line.

You do not necessarily need to follow these steps. Simpler and/or faster methods may be possible, including those that do not need any equations. **Any fully-formed, valid explanation gives full credit.**

After figuring out the simpler case, Leo realized that he could easily generalize it to the sundial mounted on the wall. He then started thinking about other ways the model and the sundial on the wall differed, and thought about the orientation of the center line, noticing that it was not perfectly horizontal, but instead slanted.

(e) **(8 points)** Based on the picture and the orientation of the center line, does the wall run perfectly East-West, or does it run Northeast-Southwest or Southeast-Northwest? Explanation needed for credit.

Assume the wall is perfectly vertical. **This part can be solved independently of the previous two parts.**