(D01) **30 Years of Exoplanets** **[90 marks]**

This problem explores some aspects of the two main methods of exoplanet detection: radial velocity and transit. Throughout this problem we shall consider a particular system of a single planet (P) in a circular orbit with radius a around a solar-type star (S). We shall refer to this system as the "SP system".

Stellar parameters

(D01.1) The V-band apparent magnitude of the star S is (7.65 ± 0.03) mag, the parallax is (20.67 ± 0.05) milliarcsecond and the bolometric correction (BC) is -0.0650 mag. Thus the star has a higher bolometric luminosity than its luminosity in the V-band. **8**

Estimate the mass of the star, M_s (in units of $M_\odot$), assuming a mass-luminosity (M–L) relation of the form $L \propto M^4$. Also estimate the uncertainty in M_s. You may need $\frac{d}{dx}\ln x = \frac{1}{x}$.

Solution:

Given

Parallax: $p = (20.67 \pm 0.05)$ milliarcsecond

V-band apparent magnitude: $m_V = (7.65 \pm 0.03)$ mag

Let:

V-band absolute magnitude: M_V

Bolometric Magnitude: M_{bol}

Bolometric Magnitude of the Sun: $M_{bol,\odot} = 4.74$ mag (given)

$$M_V = m_V + 5(\log p + 1) = 4.227\,\text{mag} \qquad 1.0$$

$$M_{bol} = M_V + BC$$

$$= m_V + 5(\log p + 1) + \text{BC} = 4.162\,\text{mag} \qquad 1.0$$

$$\text{Also} \quad M_{bol} = M_{bol,\odot} - 2.5 \log \frac{L_{bol}}{L_\odot} \qquad 1.0$$

$$\therefore \quad L_{bol} = 10^{\frac{M_{bol,\odot} - M_{bol}}{2.5}} L_\odot = 6.521 \times 10^{26}\,\text{W}$$

$$\text{Then} \quad M_s = \left(\frac{L_{bol}}{L_\odot}\right)^{1/4} M_\odot = 1.14\,M_\odot \qquad 1.0$$

Half credit lower limit	Full credit range	Half credit upper limit
$1.12\,M_\odot$	$1.13\,M_\odot$ to $1.15\,M_\odot$	$1.16\,M_\odot$

Penalty of 1 mark if intermediate step showing expression of M_{bol} is not shown.

Uncertainty estimate:

$$\Delta M_{bol} = \sqrt{(\Delta m_V)^2 + \left(\frac{5}{\ln 10}\frac{\Delta p}{p}\right)^2} \qquad 2.0$$

$$= 0.03\,\text{mag}$$

$$\text{Now,} \quad M_s = 10^{\frac{M_{bol,\odot} - M_{bol}}{10}}$$

$$\therefore \quad \Delta M_s = M_s \frac{\Delta M_{bol}}{10} \ln 10 \qquad 1.0$$

$$= 0.01\,M_\odot \qquad 1.0$$

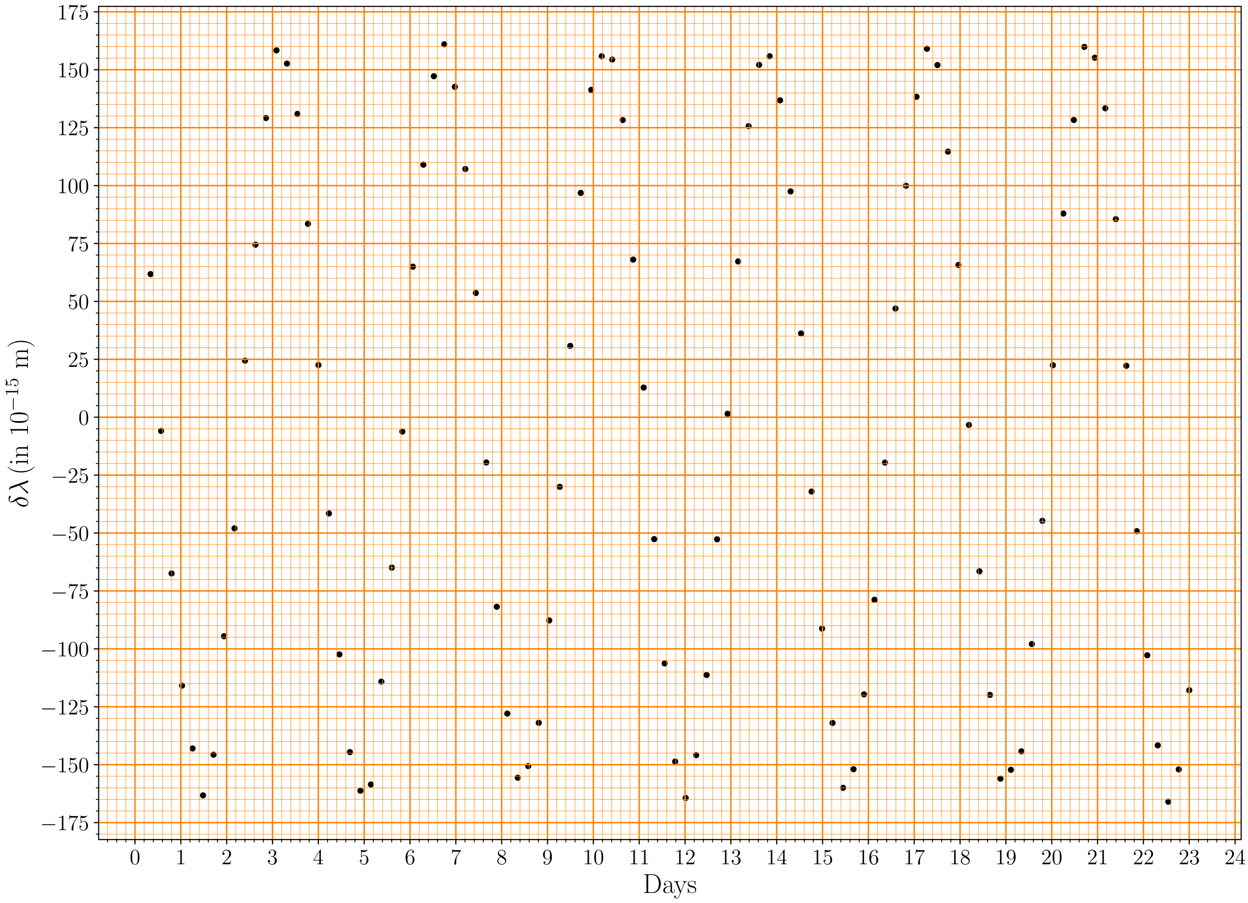
$$\boxed{M_s = (1.14 \pm 0.01)\,M_\odot}$$

Half credit lower limit	Full credit range	Half credit upper limit
	$0.008\,M_\odot$ to $0.01\,M_\odot$	

Radial Velocity method

The radial velocity method uses the Doppler shift $\delta\lambda \equiv \lambda_{\text{obs}} - \lambda_0$ between the observed wavelength λ_{obs} and the rest wavelength λ_0 of a known spectral line to detect an exoplanet and determine its characteristics.

The figure below shows the $\delta\lambda$ for the Fe I line ($\lambda_0 = 543.45 \times 10^{-9}\,\text{m}$) as a function of time as observed for the SP system.



The radial velocity semi-amplitude K is defined as $K \equiv (v_{\text{r, max}} - v_{\text{r, min}})/2$ where $v_{\text{r, max}}$ and $v_{\text{r, min}}$ are the minimum and maximum radial velocities, respectively. For a circular planetary orbit the semi-amplitude K can be written as:

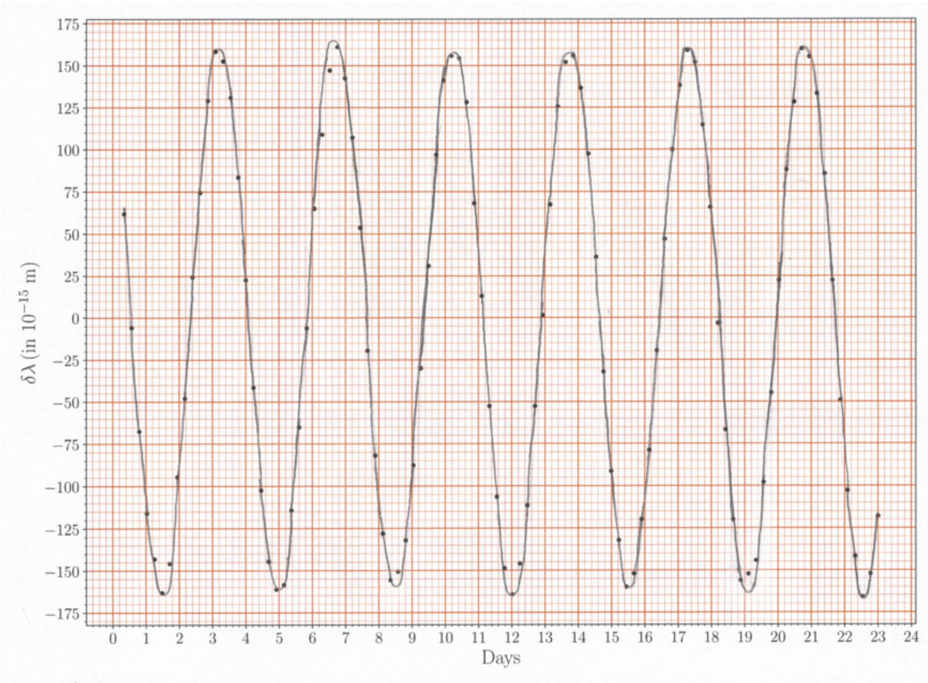
$$K = \left(\frac{2\pi G}{T}\right)^{1/3} \frac{M_{\text{p}} \sin i}{(M_{\text{p}} + M_{\text{s}})^{2/3}}$$

where T is the period, i is the inclination of the planetary orbit (angle between the normal to the orbital plane of the planet and the line of sight of the observer), M_{p} and M_{s} are the masses of the planet and the star, respectively.

(D01.2) Use the graph given in the Summary Answersheet to answer the following.

(D01.2a) Draw a smooth curve associated with the observed data shown in the graph. **2**

Solution:



2 marks: if curve is smooth

Only 1 mark: if the curve is not smooth, instead passes exactly through extrema

2.0

(D01.2b) Select appropriate points on your drawn curve and use suitable methods to determine T and K along with respective uncertainties. All data points used for the calculation of T and K must be shown in the table in the Summary Answersheet. Use the rest of the table to show your intermediate calculations, as needed, with appropriate headers. **11**

Solution:

T Determination:

Least count on time axis = 0.2 d

There are 13 zero crossings (ZC) and 6 periods.

Difference between ZC-1 and ZC-13 = (21.6 − 0.6) = 21.0 d

$$T = \frac{21}{6} = 3.5\,\mathrm{d}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
3.40 d	3.45 d to 3.55 d	3.60 d

Uncertainty in T:

$$\Delta T = \frac{1}{6}\sqrt{(0.2^2 + 0.2^2)} = 0.05\,\mathrm{d}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
0.01 d	0.02 d to 0.06 d	0.30 d

$$\boxed{T = (3.50 \pm 0.05)\,\mathrm{d}}$$

K Determination:

Serial No. of Extrema	Values of Extrema ($\times 10^{-15}$m)	$2\delta\lambda$ m	K m s^{-1}
1	-165		
2	160	325	
3	-160		
4	165	325	
5	-160		
6	155	315	
7	-165		
8	155	320	
9	-160		
10	160	320	
11	-165		
12	160	325	
13	-155		

4.0

≥ 11 points	4.0
Every less point up to 6 points	-0.5
< 6 points	0.0

Least count on $\delta\lambda$ axis $= 5 \times 10^{-15}$ m
Differences in measured extrema ($\delta\lambda$) of alternate pairs are written in table.

$$2(\delta\lambda)_{\text{Mean}} = 321.67 \times 10^{-15}\,\text{m}$$

$$(\delta\lambda)_{\text{Mean}} = 160.83 \times 10^{-15}\,\text{m}$$ 1.0

$$K = \frac{\delta\lambda}{\lambda} \times c$$ 1.0

$$= \frac{160.83 \times 10^{-15} \times 2.998 \times 10^{8}}{543.45 \times 10^{-9}}$$

$$= 88.73\,\text{m}\,\text{s}^{-1}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$86.5\,\text{m}\,\text{s}^{-1}$	$88.0\,\text{m}\,\text{s}^{-1}$ to $89.0\,\text{m}\,\text{s}^{-1}$	$89.7\,\text{m}\,\text{s}^{-1}$

Uncertainty in K:

$$\sigma_{\delta\lambda} = \sqrt{\frac{1}{N-1}\Sigma\left(\delta\lambda - (\delta\lambda)_{\text{Mean}}\right)^2}$$

$$= 2.041 \times 10^{-15}\,\text{m}$$ 1.0

$$\sigma_K = \frac{\sigma_{\delta\lambda}}{\lambda} \times c$$

$$= \frac{2.041 \times 10^{-15} \times 2.998 \times 10^{8}}{543.45 \times 10^{-9}} = 1.12\,\text{m}\,\text{s}^{-1}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	$1.0\,\text{m}\,\text{s}^{-1}$ to $1.2\,\text{m}\,\text{s}^{-1}$	

$$\boxed{K = (89 \pm 1)\,\text{m}\,\text{s}^{-1}}$$

(D01.2c) Find the minimum mass of the planet $M_{\text{p, min}}$ (in $\text{M}_\odot$), and its corresponding uncertainty, assuming $M_\text{p} \ll M_\text{s}$. **5**

Solution:
The minimum mass of the planet $M_{\mathrm{p,\,min}} \equiv M_{\mathrm{p}} \sin i$ comes from equation of K.

$$K = \left(\frac{2\pi G}{T}\right)^{1/3} \frac{M_{\mathrm{p}} \sin i}{(M_{\mathrm{p}} + M_{\mathrm{s}})^{2/3}}$$ 1.0

We assume $M_{\mathrm{p}} + M_{\mathrm{s}} \approx M_{\mathrm{s}}$. 1.0

For $M_{\mathrm{s}} = 1.14\,\mathrm{M}_\odot, T = 3.5\,\mathrm{d}$:

$$M_{\mathrm{p,\,min}} \approx K M_{\mathrm{s}}^{2/3} \left(\frac{T}{2\pi G}\right)^{1/3} = 6.927 \times 10^{-4}\,\mathrm{M}_\odot$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$6.55 \times 10^{-4}\,\mathrm{M}_\odot$	$6.75 \times 10^{-4}\,\mathrm{M}_\odot$ to $7.00 \times 10^{-4}\,\mathrm{M}_\odot$	$7.15 \times 10^{-4}\,\mathrm{M}_\odot$

Uncertainty in $M_{\mathrm{p,\,min}}$:

$$\Delta M_{\mathrm{p,\,min}} = M_{\mathrm{p,\,min}} \sqrt{\left(\frac{\Delta K}{K}\right)^2 + \left(\frac{2}{3}\frac{\Delta M_{\mathrm{s}}}{M_{\mathrm{s}}}\right)^2 + \left(\frac{1}{3}\frac{\Delta T}{T}\right)^2}$$ 1.0

$$= 0.091 \times 10^{-4}\,\mathrm{M}_\odot$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$0.07 \times 10^{-4}\,\mathrm{M}_\odot$	$0.08 \times 10^{-4}\,\mathrm{M}_\odot$ to $0.10 \times 10^{-4}\,\mathrm{M}_\odot$	$0.11 \times 10^{-4}\,\mathrm{M}_\odot$

$$\boxed{M_{\mathrm{p,\,min}} = (6.93 \pm 0.09) \times 10^{-4}\,\mathrm{M}_\odot}$$

(D01.2d) Using the value of $M_{\mathrm{p,\,min}}$ estimated in part (D01.2c), calculate the minimum value of the semi-major axis of the planet's orbit, a_{min}, in au and its uncertainty. **4**

Solution:

$$a_{\mathrm{min}} = \left(\frac{T^2 G(M_{\mathrm{s}} + M_{\mathrm{p}} \sin i)}{4\pi^2}\right)^{1/3} = \left(\frac{T^2 G(M_{\mathrm{s}} + M_{\mathrm{p,\,min}})}{4\pi^2}\right)^{1/3}$$ 1.0

$$= 0.0471\,\mathrm{au}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$0.0455\,\mathrm{au}$	$0.0465\,\mathrm{au}$ to $0.0477\,\mathrm{au}$	$0.0485\,\mathrm{au}$

Uncertainty in a_{min}:

$$\Delta a_{\mathrm{min}} = a_{\mathrm{min}} \sqrt{\left(\frac{2}{3}\frac{\Delta T}{T}\right)^2 + \left(\frac{1}{3}\right)^2 \frac{(\Delta M_{\mathrm{s}})^2 + (\Delta M_{\mathrm{p,\,min}})^2}{(M_{\mathrm{s}} + M_{\mathrm{p,\,min}})^2}}$$ 1.0

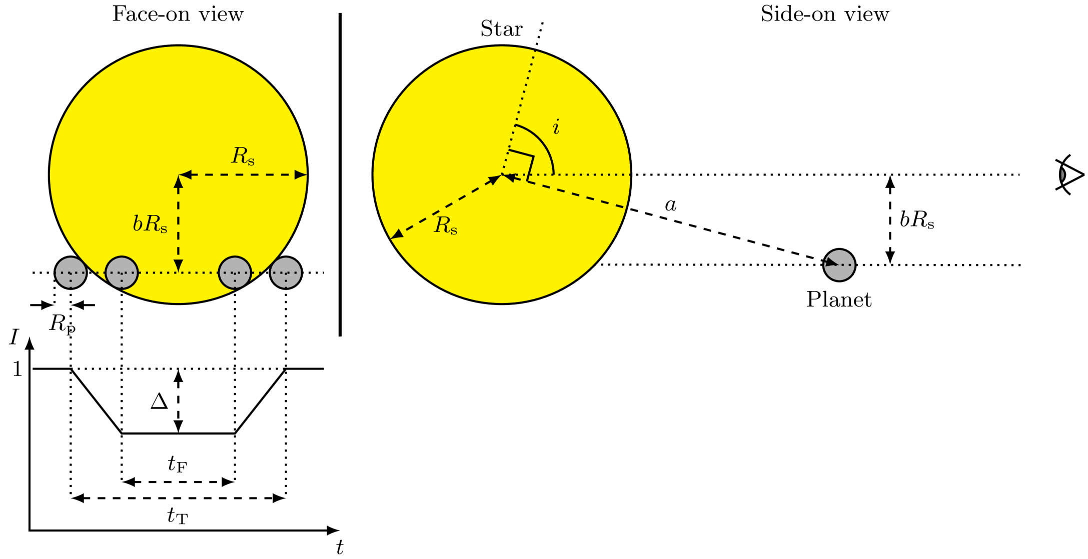
$$= 0.0005\,\mathrm{au}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$0.0001\,\mathrm{au}$	$0.0003\,\mathrm{au}$ to $0.0007\,\mathrm{au}$	$0.0009\,\mathrm{au}$

$$\boxed{a_{\mathrm{min}} = (0.0471 \pm 0.0005)\,\mathrm{au}}$$

Transit method (without limb darkening)

The schematic diagram of a planet transit (not drawn to scale) is shown below. Initially, we shall assume the stellar disk to have a uniform average intensity with some intrinsic noise due to the star itself.



The lightcurve of the normalized intensity, I, as a function of time t is shown in the schematic diagram of the transit above. The average stellar intensity outside the transit is taken as unity. The maximum decrease in the intensity is given by Δ in the normalized light curve. For a uniformly bright stellar disk, the radius of the planet, R_p, is related to Δ as

$$\left(\frac{R_p}{R_s}\right)^2 = \Delta,$$

where R_s is the radius of the star.

The total duration of transit (when part or all of the planet covers the stellar disk) is given by t_T, while t_F gives the duration when the planet is fully in front of the stellar disk. The "impact parameter" b is the projected distance between the planet and centre of the stellar disk at the mid-point of the transit, in units of the stellar radius, R_s.

For a nearly edge-on star-planet orbit, the impact parameter is given by the formula

$$b = \left[\frac{(1-\sqrt{\Delta})^2 - (t_F/t_T)^2(1+\sqrt{\Delta})^2}{1-(t_F/t_T)^2}\right]^{1/2}$$

(D01.3) For the SP system, the stellar radius is known to be $R_s = 1.20\,\mathrm{R}_\odot$, and the transit of the planet is indeed visible. Using the minimum orbital radius, a_{min}, estimated in part (D01.2d), find the minimum value, i_{min}, of the inclination angle. **3**

Solution:

$$bR_s = a\cos i \qquad \Longrightarrow i = \cos^{-1}\frac{bR_s}{a}$$ 1.0

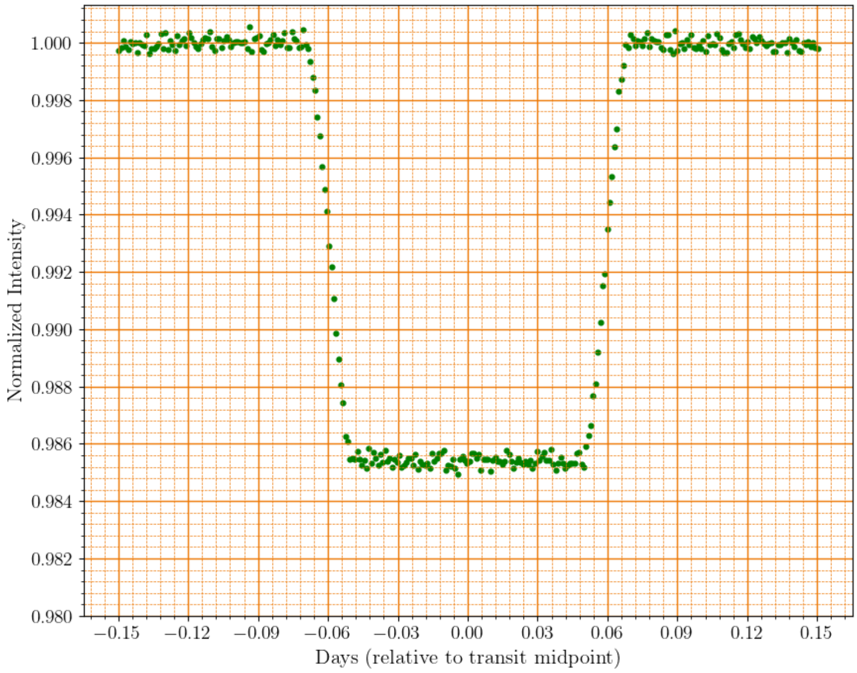
The minimum value of i corresponds to the maximum value of $b(=1)$ and the minimum value of a. Therefore,

$$i_{min} = \cos^{-1}\left(\frac{b_{max}R_s}{a_{min}}\right) = \cos^{-1}\left(\frac{R_s}{a_{min}}\right) = \cos^{-1}\left(\frac{1.20\,\mathrm{R}_\odot}{0.0471\,\mathrm{au}}\right)$$ 1.0

$$\boxed{i_{min} = 83.20^\circ}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
82.9°	83.1° to 83.3°	83.5°

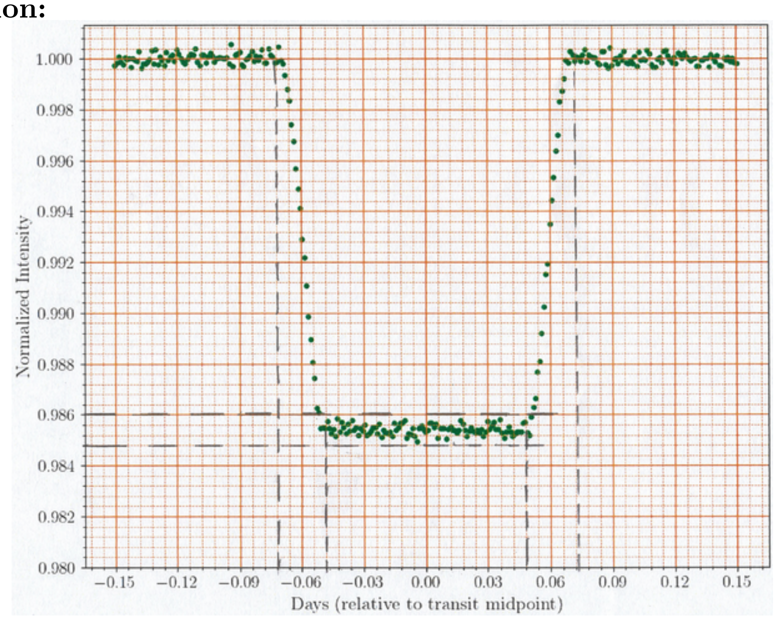
Assuming a stellar disk of uniform brightness, the transit lightcurve would look like as shown below.



(D01.4) Using the given lightcurve answer the following questions. For your reference the above lightcurve is also given in the Summary Answersheet.

(D01.4a) Estimate the values of t_T and t_F in days by marking appropriate readings on the graph. **3**

Solution:



1.0

0.5 marks each for markings on t and Normalized Intensity axis.

In the figure of the transit curve, least count on time axis $= 0.006\,$d, and on the intensity axis $= 4 \times 10^{-4}$.

The measurements on the left and right edges of the transit for t_T are $-0.072\,$d and $0.072\,$d.

The measurements on the left and right edges of the transit for t_F are

$-0.048\,\mathrm{d}$ and $0.048\,\mathrm{d}$.
Therefore,

$$\boxed{t_\mathrm{T} = 0.144\,\mathrm{d}}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	$0.132\,\mathrm{d}$ to $0.156\,\mathrm{d}$	

$$\boxed{t_\mathrm{F} = 0.096\,\mathrm{d}}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
	$0.084\,\mathrm{d}$ to $0.108\,\mathrm{d}$	

(D01.4b) Estimate the mean value of Δ, by marking appropriate readings on the graph and hence find R_p in units of $\mathrm{R}_\odot$. **2**

Solution:
The value of the dip is $(0.9860 + 0.9848)/2 = 0.9854$
$\boxed{\Delta \text{ measured from the graph is } 1 - 0.9854 = 0.0146}$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
0.0136	0.0143 to 0.0149	0.0156

$$\therefore \frac{R_\mathrm{p}}{R_\mathrm{s}} = \sqrt{\Delta} = 0.121$$
$$\implies R_\mathrm{p} = 0.121 R_\mathrm{s}$$
$$\boxed{R_\mathrm{p} = 0.145\,\mathrm{R}_\odot}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$0.14\,\mathrm{R}_\odot$	$0.144\,\mathrm{R}_\odot$ to $0.146\,\mathrm{R}_\odot$	$0.15\,\mathrm{R}_\odot$

(D01.4c) Determine the value of i in degrees assuming the orbital radius to be a_min. **2**

Solution:
$$b = \left[\frac{(1-\sqrt{\Delta})^2 - (t_\mathrm{F}/t_\mathrm{T})^2(1+\sqrt{\Delta})^2}{1-(t_\mathrm{F}/t_\mathrm{T})^2}\right]^{1/2}$$
Plugging in values of Δ, t_T, and t_F, we get $b = 0.622$ 1.0
$$bR_\mathrm{s} = a\cos i$$
Putting $a = a_\mathrm{min}$,
$$i = \cos^{-1}\left(\frac{bR_\mathrm{s}}{a_\mathrm{min}}\right)$$
$$= \cos^{-1}\left(\frac{0.622 \times 1.2\,\mathrm{R}_\odot}{0.0471\,\mathrm{au}}\right)$$
$$\boxed{i = 85.77^\circ}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
85.25°	85.50° to 86.00°	86.30°

Introducing limb darkening

So far we have assumed the stellar disk to be uniformly bright. In reality, the observed brightness of the stellar disk is not uniform due to "limb darkening" — an optical effect where the central part of the stellar disk appears brighter than the edge, or the "limb".

The limb darkening effect can be measured by the relative intensity $J(\theta) \equiv \frac{I(\theta)}{I(0)}$, where θ is the angle between the normal to the stellar surface at a point and the line joining the observer to that point, $I(\theta)$ is the observed intensity of the stellar disk at that point ($I(0)$ being the intensity at the centre of the stellar disk). For a distant observer, θ varies from $\theta = 0$ (centre of the disk) to $\theta \approx 90°$ (edge of the disk).

(D01.5) The table below gives measured $J(\theta)$ at a certain wavelength for the Sun. We shall assume that the same limb darkening profile holds for the star S.

θ	$J(\theta)$
0°	1.000
10°	0.994
15°	0.984

θ	$J(\theta)$
20°	0.971
25°	0.950
30°	0.943

θ	$J(\theta)$
40°	0.883
50°	0.794
60°	0.724

θ	$J(\theta)$
70°	0.595
80°	0.475
90°	0.312

The limb darkening profile can be modelled by a quadratic formula:

$$J(\theta) = 1 - a_1(1-\cos\theta) - a_2(1-\cos\theta)^2,$$

where a_1 and a_2 are two constants.

We shall estimate the unknown coefficients a_1 and a_2 from the given data by making a plot with suitable variables.

(D01.5a) Choose a pair of variables (x_1, y_1) which are suitable functions of θ and J, that you want to plot along x and y axes, respectively, to determine a_1 and a_2. Write the expressions for x_1 and y_1. **2**

If you need to define additional variables for additional plots, define them as (x_2, y_2), etc.

Solution:

Define $\boxed{x_1 = 1 - \cos\theta}$ and $\boxed{y_1 = \frac{J-1}{1-\cos\theta} \equiv \frac{J-1}{x_1}}$ 2.0

Then, $y_1 = -a_1 - a_2 x_1$

In this method a_1 and a_2 can be determined from the intercept and slope, respectively, of a best fit straight line to a plot of y_1 vs x_1.

(D01.5b) Tabulate the values necessary for your plots. **4**

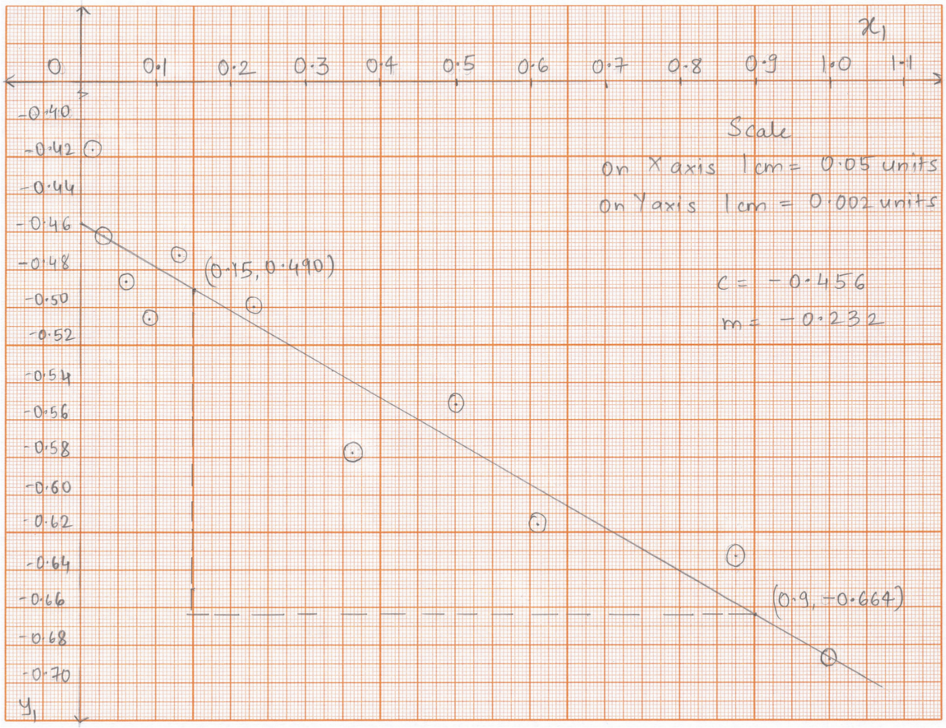
Solution:

θ	$J(\theta)$	x_1	y_1				
0°	1.000	0.000	×				
10°	0.994	0.015	−0.395				
15°	0.984	0.034	−0.470				
20°	0.971	0.060	−0.481				
20°	0.970	0.060	−0.497				
25°	0.950	0.094	−0.534				
30°	0.943	0.134	−0.425				
40°	0.883	0.234	−0.500				
40°	0.890	0.234	−0.470				
50°	0.794	0.357	−0.577				
60°	0.724	0.500	−0.552				
70°	0.595	0.658	−0.616				
80°	0.475	0.826	−0.635				
90°	0.312	1.000	−0.688				

10-11 points correct	4.0
8-9 points correct	3.0
6-7 points correct	2.0
4-5 points correct	1.0
< 4 points correct	0.0

(D01.5c) Plot the newly defined variables on the given graph paper (mark your graph as "D01.5c"). **7**

Solution:
Plot of y_1 vs x_1 with straight line fit:



Label for both x_1 and y_1 axis mentioned	0.5
Either scale mentioned or ticks marked on both axes	1.0
More than 75% of the graph paper used	0.5
11 points marked correctly in the plot	3.0
9 to 10 points marked correctly in the plot	2.5
7 to 8 points marked correctly in the plot	2.0
5 to 6 points marked correctly in the plot	1.0
<5 points in the plot	0
Drawing best fit line	2.0

(D01.5d) Obtain a_1 and a_2 from the plot. Uncertainties on the values are not needed. **7**

Solution:
Slope of best fit line $= -0.232$; Intercept $= -0.456$

$$\boxed{\Longrightarrow \quad a_1 = 0.46, \quad a_2 = 0.23}$$

Procedure of calculation of slope on graph	2.0
Procedure of calculation of intercept on graph	1.0

Correct value of slope		1.0
Half credit lower limit	Full credit range	Half credit upper limit
-0.250	-0.240 to -0.220	-0.210

Correct value of intercept		1.0
Half credit lower limit	Full credit range	Half credit upper limit
-0.470	-0.465 to -0.445	-0.440

Inferring values of a_1 and a_2 from slope and intercept	2.0

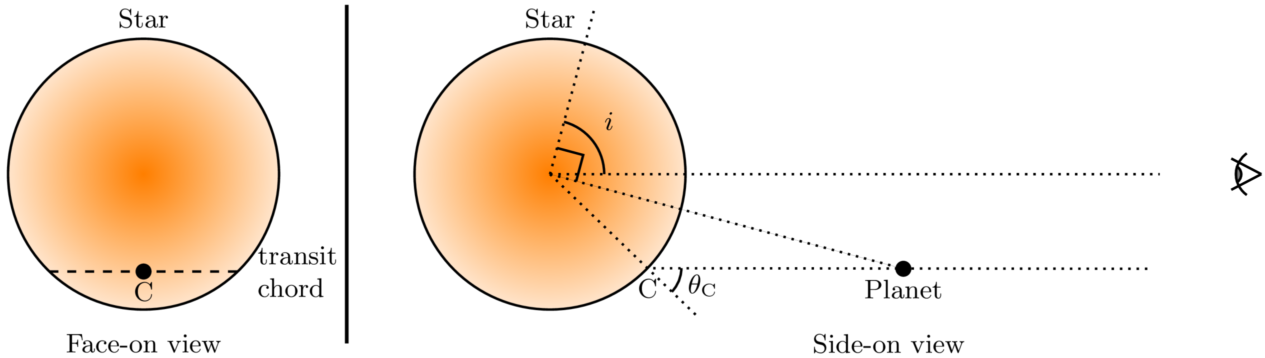
Transit in the presence of limb darkening

Now, we consider planetary transits across a limb darkened stellar disk. In the presence of limb darkening, which we shall model by the quadratic formula of $J(\theta)$ given above, the average observed intensity of the entire stellar disk (without any transit), $\langle I \rangle$, is given by:

$$\langle I \rangle = \left(1 - \frac{a_1}{3} - \frac{a_2}{6}\right) I(0)$$

Further, the dip in the light caused by the transiting planet now depends not only on the relative size of the planet and the star, $\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)$, but also on the intensity profile of the stellar disk along the transit chord, which in turn, depends on the angle of inclination, i.

The schematic diagram below (not drawn to scale) shows the configuration. Note that the brighter part of the star is shown in a darker shade, while the planet is shown as a black dot.



Here the relation between $\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)$ and the measured Δ from the light curve is

$$\Delta = \frac{I(\theta_\mathrm{C})}{\langle I \rangle} \left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)^2,$$

where $I(\theta_\mathrm{C})$ is the intensity of the stellar disk at the midpoint of the transit chord (point C in the figure above), θ_C being the angle between the line of sight and the normal to the surface at that point. From the above it is obvious that for a given star, the same value of Δ can be produced by many combinations of the planet size, R_p, and the inclination angle i.

(D01.6) It is possible to uniquely determine both R_p and i by using data from transit lightcurves at two wavelengths, say, λ_B (blue) and λ_R (red). The limb darkening coefficients for these two wavelengths are given below:

Wavelength	a_1	a_2
λ_B	0.82	0.05
λ_R	0.24	0.20

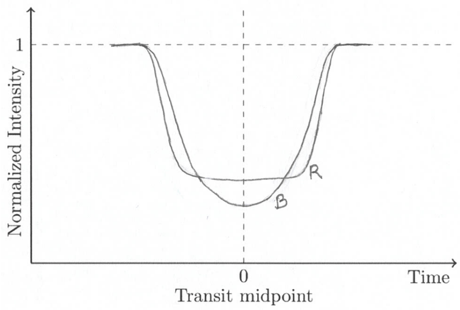
(D01.6a) Choose the correct statement among the following that describes the relation between the maximum depth of the transit (Δ) for λ_B and the inclination angle (i) of the orbit and tick (✓) it in the Summary Answersheet. **2**

(A) Δ increases with decreasing i.
(B) Δ decreases with decreasing i.
(C) Δ is independent of i.

Solution:
As i decreases, the transit chord moves closer to the limb and the contribution of the midpoint of the transit to the dip in the average brightness reduces. Therefore, Δ decreases with decreasing i.
Correct option: $\boxed{B}$ 2.0

(D01.6b) The maximum depth of the transit (Δ) for the "SP system" was measured to be 0.0182 and 0.0159 for λ_B and λ_R, respectively. **4**
Draw schematic transit light curves for both λ_B and λ_R on the given grid and label the curves by "B" and "R" respectively. Assume that the total transit duration is same for both wavelengths. The curves need not be to scale, but should represent the shapes of the light curves correctly.

Solution:
Since the limb darkening effect is stronger for λ_B, it causes *lesser* reduction in the total light at the ingress and egress and more reduction at the transit centre, than for λ_R. Therefore, the curves must intersect.



Credit points:

The curves are symmetric about transit midpoint	0.5
t_T is same for both	0.5
Blue is deeper than red	1.0
The curves intersect	1.0
Blue curve is more rounded than red	1.0

(D01.7) We shall use a graphical method to find the values of R_p and i for the SP system using measurements of Δ at λ_B and λ_R.

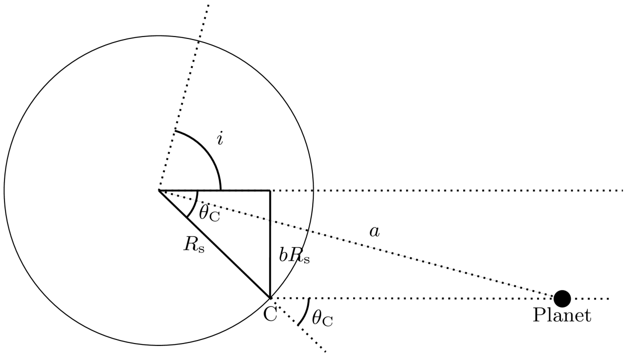
(D01.7a) Write an appropriate expression connecting the relevant variables that are to be plotted. (Hint: You may consider i or b, and R_p among the relevant variables.) **6**

Solution:

$$\Delta = \frac{I(\theta_\mathrm{C})}{\langle I \rangle}\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)^2$$

$$\implies \left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right)^2 = \frac{\Delta\langle I \rangle}{I(\theta_\mathrm{C})}$$ 1.0

Here, $I(\theta_\mathrm{C})$ is a function of i. Further, $\langle I \rangle = \left(1 - \frac{a_1}{3} - \frac{a_2}{6}\right) I(0)$.



From the figure,

$$\sin\theta_\mathrm{C} = \frac{bR_\mathrm{s}}{R_\mathrm{s}} = b = \frac{a\cos i}{R_\mathrm{s}}$$

$$\implies \cos\theta_\mathrm{C} = \sqrt{1-b^2} = \sqrt{1-\left(\frac{a\cos i}{R_\mathrm{s}}\right)^2}$$ 2.0

Therefore,

$$\begin{aligned} I(\theta_\mathrm{C}) &= J(\theta_\mathrm{C}) I(0) \\ &= \left[1 - a_1(1-\cos\theta_\mathrm{C}) - a_2(1-\cos\theta_\mathrm{C})^2\right] I(0) \\ &= \left[1 - a_1\left(1-\sqrt{1-b^2}\right) - a_2\left(1-\sqrt{1-b^2}\right)^2\right] I(0) \end{aligned}$$ 1.0

Thus,

$$\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right) = \left[\frac{\Delta\langle I \rangle}{I(\theta_\mathrm{C})}\right]^{1/2}$$

$$\boxed{\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right) = \left[\frac{\Delta\left(1-\frac{a_1}{3}-\frac{a_2}{6}\right)}{1 - a_1\left(1-\sqrt{1-b^2}\right) - a_2\left(1-\sqrt{1-b^2}\right)^2}\right]^{1/2}}$$ 2.0

$$\boxed{\left(\frac{R_\mathrm{p}}{R_\mathrm{s}}\right) = \left[\frac{\Delta\left(1-\frac{a_1}{3}-\frac{a_2}{6}\right)}{1 - a_1\left(1-\sqrt{1-\left(\frac{a\cos i}{R_\mathrm{s}}\right)^2}\right) - a_2\left(1-\sqrt{1-\left(\frac{a\cos i}{R_\mathrm{s}}\right)^2}\right)^2}\right]^{1/2}}$$ (2.0)

(D01.7b) Tabulate the appropriate quantities that are to be plotted. **5**

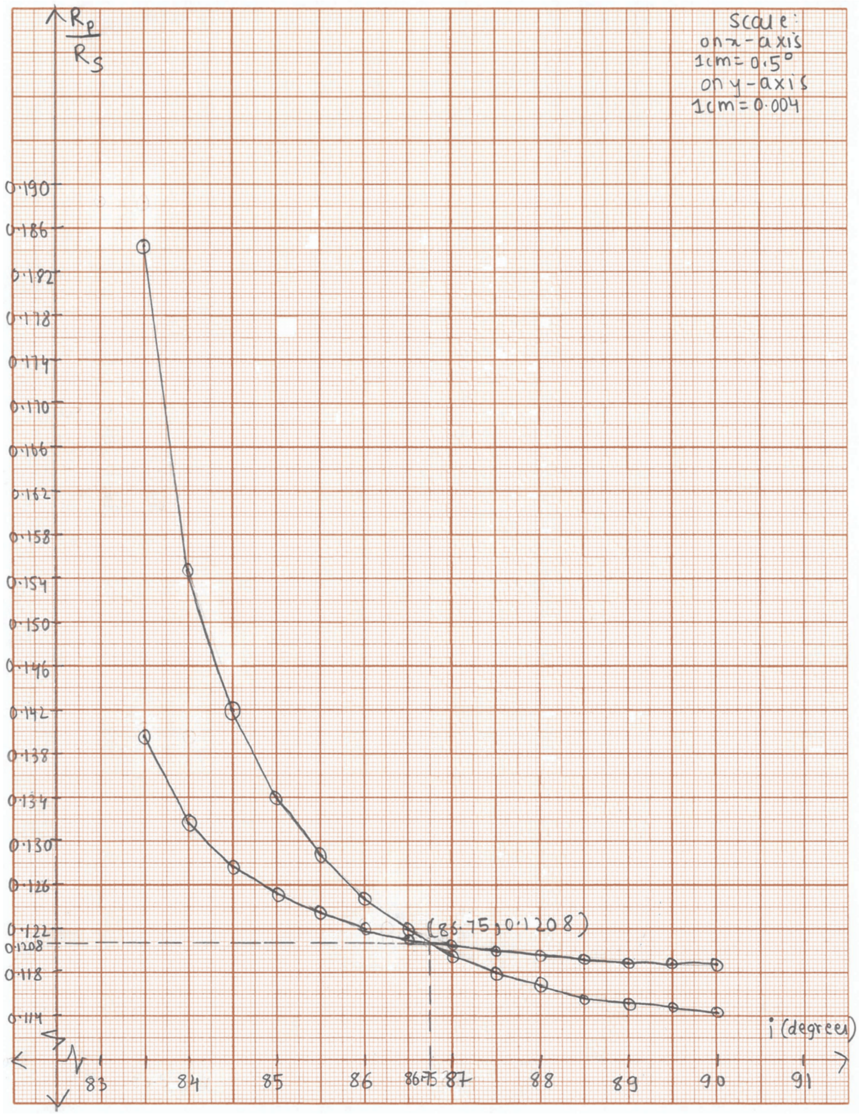
Solution:

i (°)	b	$(R_p/R_s)_{\lambda_B}$	$(R_p/R_s)_{\lambda_R}$
83.5	0.96	0.1842	0.1395
84.0	0.89	0.1547	0.1316
84.5	0.81	0.1419	0.1276
85.0	0.74	0.1341	0.1251
85.5	0.66	0.1287	0.1234
86.0	0.59	0.1248	0.1221
86.5	0.52	0.1218	0.1211
87.0	0.44	0.1196	0.1204
87.5	0.37	0.1178	0.1198
88.0	0.30	0.1165	0.1194
88.5	0.22	0.1155	0.1191
89.0	0.15	0.1149	0.1189
89.5	0.07	0.1145	0.1188
90.0	0.00	0.1143	0.1187

Number of points for each wavelength ≥ 7	5.0
Number of points for each wavelength $= 6$	4.0
Number of points for each wavelength $= 5$	3.0
Number of points for each wavelength $= 4$	2.0
Number of points for each wavelength $= 3$	1.0
Number of points for each wavelength < 3	0.0

(D01.7c) Draw a suitable graph and mark it as "D01.7c". **7**

Solution:
One can plot the full range ($i_{\min} \leq i \leq 90°$):



However, it is enough to plot around the point of intersection (can be deduced from the table above), to get a better resolution for the coordinates of the intersection point.

Label for both i and $\left(\frac{R_p}{R_s}\right)$ axis mentioned		0.5
Either scale mentioned or ticks marked on both axes		1.0
More than 75% of the graph paper used		0.5
Maximum interval in i between successive points on each curve $\leq 0.5°$ (2×0.5)		1.0
≥ 8 points marked correctly in the plot		2.0
6 to 7 points marked correctly in the plot	1.5	
4 to 5 points marked correctly in the plot	1.0	
≤ 3 points marked correctly in the plot	0.0	
Drawing smooth curves through the points		2.0

(D01.7d) Estimate the values of R_p (in $R_\odot$) and i (in degrees) from the graph. **4**

Solution:

Intersection point of the graphs of $\left(\frac{R_{\mathrm{p}}}{R_{\mathrm{s}}}\right)$ vs i for λ_{B} and λ_{R} will give the values of both quantities. 1.0

Values deduced from either graph:

$$\left(\frac{R_{\mathrm{p}}}{R_{\mathrm{s}}}\right) = 0.1208$$ 1.0

$$\implies R_{\mathrm{p}} = 0.1208 \times 1.2\,\mathrm{R}_{\odot}$$

$$\boxed{R_{\mathrm{p}} = 0.145\,\mathrm{R}_{\odot}}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
$0.143\,\mathrm{R}_{\odot}$	$0.144\,\mathrm{R}_{\odot}$ to $0.146\,\mathrm{R}_{\odot}$	$0.147\,\mathrm{R}_{\odot}$

and

$$\boxed{i = 86.75^{\circ}}$$ 1.0

Half credit lower limit	Full credit range	Half credit upper limit
86.72°	86.73° to 86.77°	86.78°

(D01.8) Based on the results obtained in this problem, indicate whether the planet P is "ROCKY" or "GASEOUS" by ticking (✓) the appropriate box in the Summary Answersheet. **2**

Solution:

$$\text{Estimated mass of the planet P } M_{\mathrm{p}} = 6.93 \times 10^{-4}\,\mathrm{M}_{\odot} = 1.389 \times 10^{27}\,\mathrm{kg}$$

$$\text{Estimated radius of the planet P } R_{\mathrm{p}} = 0.145\,\mathrm{R}_{\odot} = 1.009 \times 10^{8}\,\mathrm{m}$$

$$\therefore \text{Average density of the planet} = 320\,\mathrm{kg\,m^{-3}}$$

The average density of the planet P is much less than that of Jupiter (a known gaseous planet). Hence the planet P is $\boxed{\text{GASEOUS}}$.

Credit will be given only if answer is consistent with the derived values of M_{p} and R_{p}.