IOAA 2023 General Marking Scheme

Using incorrect physical concept (despite correct answers)	No points given
Giving correct answer without detailed calculation	Deduct 50% of the marks for that part
Minor mistakes in the calculations, e.g., wrong signs, symbols, substitutions	Deduct 20% of the marks for that part
Units missing from final answers	Deduct 0.5 pts
Too few or too many significant figures in the final answer	Deduct 0.5 pts
Error resulting from another error in an earlier part for which the student already lost marks, if the answer is physically reasonable.	Full points (i.e., no deductions)
Error resulting from another error in an earlier part, where the student should have realised the answer was physically unreasonable.	Deduct 20% of the marks for that part

For example, if due to an error in an earlier part, the student calculates the mass of a star as 2.5×10^{30} kg *instead of* 2×10^{30} kg*, they will only lose marks for the earlier part. However, if, for the same reason, a student calculates the mass as* 2×10^{25} kg*, they should realize this is wrong (a few times the Earth's mass) and thus should lose some marks for this part as well.*

Table of Constants

Fundamental constants

Speed of light in vacuum	c	=	$2.998 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Planck constant	h	=	$6.626 \times 10^{-34}\ \mathrm{J\,s}$
Boltzmann constant	k_B	=	$1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	$5.670 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$
Elementary charge	e	=	$1.602 \times 10^{-19}\ \mathrm{C}$
Universal gravitational constant	G	=	$6.674 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Universal electric constant	ϵ_0	=	$8.854 \times 10^{-12}\ \mathrm{m^{-3}kg^{-1}s^4A^2}$
Universal gas constant	R	=	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_A	=	$6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
Wien's displacement constant	$b = \lambda_m T$	=	$2.898 \times 10^{-3}\ \mathrm{m\,K}$
Mass of electron	m_e	=	$9.109 \times 10^{-31}\ \mathrm{kg}$
Mass of proton	m_p	=	$1.673 \times 10^{-27}\ \mathrm{kg}$
Mass of neutron	m_n	=	$1.675 \times 10^{-27}\ \mathrm{kg}$
Mass of Helium nucleus	m_He	=	$6.645 \times 10^{-27}\ \mathrm{kg}$
Atomic mass unit (a.m.u., Dalton)		=	$1.661 \times 10^{-27}\ \mathrm{kg}$

Astronomical data

Hubble constant	H_0	=	$70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$
North Ecliptic Pole (J2000.0)	$(\alpha_\mathrm{E},\ \delta_\mathrm{E})$		$(18^\mathrm{h}00^\mathrm{m}00^\mathrm{s}, +66^\circ 33' 39'')$
North Galactic Pole (J2000.0)	$(\alpha_\mathrm{G},\ \delta_\mathrm{G})$		$(12^\mathrm{h}51^\mathrm{m}26^\mathrm{s}, +27^\circ 07' 42'')$
1 jansky	1 Jy	=	$10^{-26}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$
1 parsec	1 pc	=	$3.086 \times 10^{16}\ \mathrm{m}$ 206 265 au 3.262 ly
1 astronomical unit (au)	1 au	=	$1.496 \times 10^{11}\ \mathrm{m}$
1 sidereal day	T_SD	=	23.93444 h $23^\mathrm{h}56^\mathrm{m}04^\mathrm{s}$
1 tropical year		=	365.2422 solar days
1 sidereal year		=	365.2564 solar days

Gauss's formulae

Spherical law of cosines: $\cos a = \cos b \cos c + \sin b \sin c \cos A$

Spherical law of sines: $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$

Approximations

$(1+x)^n \approx 1 + nx$

$(1+x)(1+y) \approx 1 + x + y$ if $x \ll 1$ and $y \ll 1$

The Sun

Solar luminosity	$L_\odot$	$=$	3.826×10^{26} W
Apparent angular diameter of Sun	$\theta_\odot$	$=$	$32'$
Effective temperature of Sun	$T_{\mathrm{eff},\odot}$	$=$	5778 K
Apparent visual magnitude		$=$	-26.75
Absolute visual magnitude		$=$	+4.82
Apparent bolometric magnitude		$=$	-26.83
Absolute bolometric magnitude		$=$	+4.74
Distance of the Sun from the Galactic centre		$\approx$	8 kpc

The Earth and Moon

Obliquity of the ecliptic (Earth)	ϵ	$=$	$23.5'$
Platonic year (period of precession of Earth's axis)		$=$	25 765 years
Apparent visual magnitude of full Moon		$=$	-12.74
Apparent angular diameter of Moon	θ_{L}	$=$	$31'$
Inclination of the lunar orbit to the ecliptic		$=$	$05°08'43''$
Inclination of the lunar equator to its orbital plane		$=$	6.687°
Lunar sidereal month	T_{SL}	$=$	27.321661 d 655.71986 h
Synodic month		$=$	29.530589 d
Tropical month		$=$	27.321582 d
Anomalistic month		$=$	27.554550 d
Draconic month		$=$	27.212221 d

The Solar System

Object	**Mean radius** [km]	**Mass** [kg]	**Semimajor axis** [au]	**Eccentricity**
Sun	695 700	1.988×10^{30}	—	—
Mercury	2 440	3.301×10^{23}	0.387	0.206
Venus	6 052	4.867×10^{24}	0.723	0.007
Earth	6 378	5.972×10^{24}	1.000	0.016 710
Moon	1 737	7.346×10^{22}	3.844×10^5 km	0.054 900 (range $0.026 - 0.077$)
Mars	3 390	6.417×10^{23}	1.524	0.093
Jupiter	69 911	1.898×10^{27}	5.203	0.048
Saturn	58 232	5.683×10^{26}	9.537	0.054
Uranus	25 362	8.681×10^{25}	19.189	0.047
Neptune	24 622	1.024×10^{26}	30.070	0.009

Theory: instructions

- **Do not touch envelopes until the start of the examination.**
- The theoretical examination lasts for 5 hours and is worth a total of 250 marks.
- There are **Answer Sheets** for carrying out detailed work and **Working Sheets** for rough work, which are already marked with your student code and question number.
- ***Use only the answer sheets for a particular question for your answer. Please write only on the printed side of the sheet. Do not use the reverse side.*** If you have written something on any sheet which you do not want to be evaluated, cross it out.
- Use as many mathematical expressions as you think may help the evaluator to better understand your solutions. The evaluator may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
- You are not allowed to leave your work desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, etc.), please draw the attention of the supervisor.
- The beginning and end of the examination will be indicated by the supervisor. The remaining time will be displayed on a clock.
- At the end of the examination you must stop writing immediately. Put everything back in the envelope and leave it on the table.
- Once all envelopes are collected, your student guide will escort you out of the examination room.
- A list of constants and useful relations are included in the envelope.

Theory 1: 'Neptune'

Given that Neptune will be at opposition on 21 September 2024, calculate in which year Neptune was last at opposition near the time of the northern-hemisphere spring equinox. Assume that the orbits of Earth and Neptune are circular.

(5 points)

Solution

Using Kepler's Third law and the semi-major axis ($a = 30.070$ au) of the orbit from the table of constants, the sidereal period of Neptune's orbit is:

$$P_\mathrm{N} = (a^3)^{\frac{1}{2}} = \sqrt{27189.4} = 164.89\,\text{years} \qquad \text{(1 point)}$$

The rest of the calculation can be done in (at least) two ways:

(1) 'day counting'/'date drift' solution:

Since Neptune is an outer planet (relative to Earth), the synodic period S in years is given by:

$$\frac{1}{S} = \frac{1}{1\,\text{year}} - \frac{1}{P_\mathrm{N}} \qquad \text{(1 point)}$$

$$= 1 - \frac{1}{164.89} = 1 - 0.006065 = 0.99394$$

$$\implies S = 1/0.99394 = 1.006102\,\text{years} \qquad \text{(1 point)}$$

$$\implies 1.006102 \times 365.2422 = 367.4708\,\text{solar days}$$

Therefore the date of opposition drifts by:

$$D = 367.4708 - 365.2422 = 2.2286\,\text{days/year} \qquad \text{(1 point)}$$

The approximate date of the northern spring equinox is 20 March. The number of days between 21 September and 20 March = 185 days, therefore year of desired opposition is:

$$2024 - (185/D) = 2024 - 83 = 1941 \qquad \text{(1 point)}$$

If the student takes 21 March as the spring equinox, they get $184/D = 82.5\,\text{yr} \implies 1942$.

(2) 'angular drift' solution (more accurate)

As before, the synodic period of Neptune can be derived as:

$$\frac{1}{S} = \frac{1}{1\,\text{year}} - \frac{1}{P_\mathrm{N}} \qquad \text{(1 point)}$$

$$= 1 - \frac{1}{164.89} = \frac{163.89}{164.89} \implies S = \frac{164.89}{163.89}\,\text{years}$$

Therefore the ecliptic longitude of Neptune at opposition drifts by $(360^\circ/163.89)$/year. We want to find how many years it takes for the longitude of opposition to move by 180°, i.e. for what t:

$$t \times \frac{360^\circ}{163.89} = 180^\circ. \qquad \text{(2 points)}$$

$$\implies t = 163.89/2 = 81.95\,\text{years} \implies 2024 - 82 = 1942 \qquad \text{(1 point)}$$

We accept 1941 or 1942 for full points for calculations using the assumptions in the question. If the student uses some other method which is conceptually correct and results in 1943 (the true answer) they should also get full points.

Table of spring equinoxes and oppositions of Neptune:

Year	Equinox (UT)	Opposition (UT)	Coordinates of Neptune	ΔT [days]
1940	Mar 20 18:42	Mar 14 21:08	11h40m $+3°28'$	5.9
1941	Mar 21 00:20	Mar 17 07:40	11h49m $+2°39'$	3.7
1942	Mar 21 06:11	Mar 19 18:12	11h57m $+1°50'$	1.5
1943	Mar 21 12:03	Mar 22 04:51	12h04m $+1°00'$	-0.7
1944	Mar 20 17:49	Mar 23 15:29	12h13m $+0°11'$	-2.9
2024	Mar 20 03:06	Sep 21 00:16	23h55m $+1°56'$	-184.9

Taking into account all effects, the last opposition closest to the spring equinox was actually in 1943. 1942 results from the assumptions made in the question.

Theory 2: 'Magnetic field'

An emission line of wavelength $\lambda = 600$ nm was observed in the spectrum of a white dwarf. Assuming that it originates from the interaction of an electron with a magnetic field,

(a) calculate the magnetic flux density of the field;

(b) estimate the wavelength of another spectral line, the discovery of which could confirm that the lines originate from particles of a plasma interacting with the magnetic field.

(5 points)

Solution

(a) In a magnetic field, a charged particle moves along a circular path defined by the equality of centrifugal and magnetic forces:

$$mv^2/r = evB, \qquad \text{(1 point)}$$

where m is the mass, v velocity, r radius of the circle, e charge of the particle, and B magnetic flux density.

For circular motion, $v = 2\pi r/T$, therefore $T = 2\pi m/eB$. The charged particle moving in harmonic motion (i.e. along the circular path with constant velocity) emits a wave of wavelength $\lambda = cT = 2\pi mc/eB$ and thus $B = 2\pi mc/e\lambda$. (1 point)

Substituting the numerical values, including the mass and charge of the electron:

$$B = (2\pi \times 9.109 \times 10^{-31} \times 2.998 \times 10^{8})/(1.602 \times 10^{-19} \times 6 \times 10^{-7}) \approx 2 \times 10^{4}\ \text{T}. \quad \text{(1 point)}$$

Since λ is given to 1 s.f., the correct answer is 20 kT, however accept 17.9 kT or 18 kT. More than 3 s.f. in the final answer loses points.

(b) In the plasma, besides electrons, only protons will be present in large quantities; protons will emit energy at a wavelength larger in proportion to the mass ratio, i.e. 1836× larger (anything within 1800–2000× $\implies$ λ =1.08–1.20 mm is acceptable).

(2 points)

Theory 3: 'Microlensing'

A faint subdwarf star ($I = 20.4\,\mathrm{mag}$) in the Galactic bulge was observed to brighten to $I' = 15.2\,\mathrm{mag}$ as a result of gravitational microlensing, allowing a high-resolution spectrum to be obtained with the UVES spectrograph on the Very Large Telescope (mirror diameter 8.2 m).

Estimate the diameter of the telescope needed to obtain a spectrum of the same quality with the same instrument and exposure time for this star at its normal apparent brightness.

(5 points)

Solution

Let F be the unmagnified flux of the star. During gravitational microlensing, the apparent flux is magnified by a factor of A, thus using the formula relating magnitude to flux:

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right), \qquad \text{(1 point)}$$

$$I - I' = -2.5 \log_{10}\left(\frac{F}{AF}\right) = 2.5 \log A \qquad \text{(1 point)}$$

and so

$$A = 10^{2.08} \approx 120. \qquad \text{(1 point)}$$

Let D be the effective mirror diameter of a telescope which would collect the same number of photons in unit time from the unbrightened star as VLT from the brightened star. We have:

$$FD^2 = AFD_{\mathrm{VLT}}^2, \qquad \text{(1 point)}$$

therefore:

$$D = \sqrt{A}\, D_{\mathrm{VLT}} \approx 90\,\mathrm{m}. \qquad \text{(1 point)}$$

Theory 4: 'Europa'

(a) Assuming that the ice covering the ocean on Jupiter's moon Europa is 6 km thick, that the surface temperature on the night side of Europa is 100 K and that the temperature at the ice-water boundary is 273 K, calculate the total power corresponding to the heat emitted from the interior of this moon.

(b) On Earth, the geothermal heat flux measured at the surface is $70 \times 10^{-3}\,\mathrm{Wm^{-2}}$ and originates mainly from radioactive decay. Is the heat emanating from the interior of Europa more likely to come from radioactive decay or tidal forces? (Select the correct answer on the answer sheet and show your working.)

(10 points)

Notes: the heat passing through a wall with a surface S and thickness d in time t is described by the formula:

$$Q = \lambda S \Delta T t/d,$$

where λ stands for thermal conductivity and ΔT for the temperature difference.

The thermal conductivity of ice $\lambda = 3\,\mathrm{Wm^{-1}K^{-1}}$. The mass and radius of Europa are 4.8×10^{22} kg and 1561 km.

Solution

(a) From the formula $Q = \lambda S \Delta T t/d$, calculate the power P of the heat flowing through a unit of the surface of ice:

$$P = Q/t = \lambda S \Delta T/d. \qquad \text{(1 point)}$$

Approximating the ice crust of Europa as a 'wall', i.e. that the upper and lower surfaces are of equal area S, we obtain the power per unit surface area: (1 point)

$$P/S = \lambda \Delta T/d.$$

Substituting the data we obtain

$$P/S = 86.5 \times 10^{-3}\,\mathrm{Wm^{-2}} \approx 87\,\mathrm{mWm^{-2}}, \qquad \text{(1 point)}$$

similar to the value given for the Earth.
The total power emitted inside Europa is therefore equal to

$$4\pi R_{Eu}^2\,(P/S) = 2.65 \times 10^{12}\,\mathrm{W}. \qquad \text{(2 points)}$$

(b) The total power emitted inside the Earth is equal to

$$4\pi R_{\oplus}^2 \times 70 \times 10^{-3}\,\mathrm{Wm^{-2}} = 36 \times 10^{12}\,\mathrm{W}, \qquad \text{(1 point)}$$

which is about 13.5× larger than the value for Europa. However, the mass ratio is 124. For the heat in Europa's interior to have a purely radioactive origin, the matter on Europa would have to contain an order of magnitude more radioactive elements per unit mass, which excludes this explanation. (3 points)

Thus the answer must be tidal forces. (1 point)

Theory 5: 'Dark Energy'

Observations indicate that the expansion of the Universe is accelerating. Fluctuations of the cosmic microwave background favour a flat (Euclidean) geometry, in which the total mass density (i.e. density of matter and equivalent mass density of all forms of energy) should be equal to the so-called critical density:

$$\rho_{\rm cr} = \frac{3\,H_0^2}{8\,\pi\,G},$$

where H_0 is the present value of the Hubble constant. However, the total density of matter (luminous and dark) is estimated at

$$\rho_{\rm m,0} \approx 2.8\cdot 10^{-27}\,{\rm kg\,m^{-3}}\,.$$

To resolve this discrepancy, the standard cosmological model assumes that the Universe is filled with a mysterious 'dark energy' of constant energy density E_Λ.

Determine the value of E_Λ and calculate for which redshift in the past the energy density equivalent to matter was equal to the density of dark energy. Neglect the contribution of electromagnetic radiation.

(12 points)

Solution

Substituting the values of H_0 and G from the table of constants,

$$\rho_{\rm cr} = 9.202\times 10^{-27}\,{\rm kg\,m^{-3}}. \qquad \text{(1 point)}$$

In flat geometry we have:

$$\rho_{m,0} + \frac{E_\Lambda}{c^2} = \rho_{\rm cr}\,. \qquad \text{(2 points)}$$

Hence E_Λ is given by:

$$E_\Lambda = [\rho_{\rm cr} - \rho_{m,0}]\,c^2 \approx 5.756\times 10^{-10}\,{\rm J\,m^{-3}}. \qquad \text{(1 points)}$$

The linear scale of the Universe, a, is related to the cosmological redshift:

$$a(z) = a_0/(1+z)\,. \qquad \text{(2 points)}$$

Thus, the matter density, ρ_m, increases with redshift:

$$\rho_m(z) = \rho_{m,0}\,(1+z)^3. \qquad \text{(2 points)}$$

We substitute the matter density, ρ_m by the energy density, E_m, using the relationship $E = mc^2$

$$E_m(z) = \rho_m(z)\,c^2 = \rho_{m,0}\,(1+z)^3\,c^2. \qquad \text{(2 points)}$$

We finally get

$$z_{eq} = \left[\frac{E_\Lambda}{\rho_{m,0}\,c^2}\right]^{1/3} - 1 \approx 0.32\,. \qquad \text{(2 points)}$$

Theory 6: 'Bolometer'

The entrance cavity of a particular bolometer is a cone with an opening angle of $30°$, the surface of which has an energy absorption coefficient of $a = 0.99$. Assume that there is no scattering of the incident radiation on the walls of the cavity, only multiple specular reflections. The bolometer is connected to a cooler which keeps the bolometer cavity surface at practically 0 K temperature. The instrument is orbiting at 2 au from the Sun and is pointed directly at the centre of the Solar disk.

Calculate the temperature of a black body which would radiate the same amount of energy from a unit surface area as the bolometer opening does.

Note: the opening angle is defined as twice the angle between the axis of the cone and its generatrix.

(13 points)

Solution

From the table of constants, the Solar luminosity $L_\odot = 3.826\times10^{26}\,\mathrm{W}$ and $1\,\mathrm{au} = 1.496\times10^{11}\,\mathrm{m}$, therefore the distance to the bolometer is:

$$r = 2\,\mathrm{au} = 2.992\times10^{11}\,\mathrm{m}$$

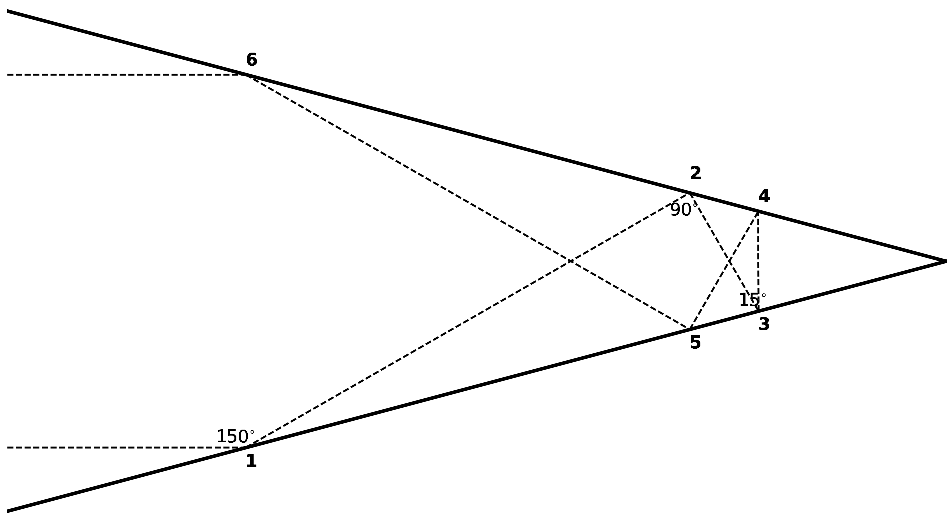
and the surface area A of a sphere of that radius is:

$$A_{\mathrm{sphere}} = 4\pi r^2 = 1.125\times10^{24}\,\mathrm{m}^2$$

and the incident flux is:

$$F(r) = L_\odot/A_{\mathrm{sphere}} = 340.1\,\mathrm{W\,m^{-2}} \qquad \text{(3 points)}$$

The incoming radiation can be assumed to be initially parallel to the axis of the cone. As the rays hit the surface they are reflected, after the first reflection the rays are travelling at 30° to their original path (2 × 15°). They next meet the surface at 30° + 15° = 45° and are reflected by 90°, and so on with each reflection, such that after $N = 6$ reflections the ray will be sent back out of the aperture as shown in the figure below. These exiting rays are what we are interested in; all the energy entering the bolometer itself is removed by the 100% efficient cooler.



(*conceptually difficult:* 4 points)

Thus the fraction of energy leaving the bolometer will be given by:

$$S = (1-a)^N = 0.01^6 = 10^{-12}. \qquad \text{(2 points)}$$

Treating the opening as if it were a black body radiator, we can apply the Stefan-Boltzmann law:

$$\Phi = \sigma T^4 \qquad \text{(1 point)}$$

$$\implies T = (\Phi/\sigma)^{0.25},$$

and

$$\Phi = F(r)\cdot S = 340.1\times10^{-12}\,\mathrm{Wm^{-2}} \qquad \text{(2 points)}$$

From the table of constants: $\sigma = 5.670\times10^{-8}\,\mathrm{Wm^{-2}K^{-4}}$, therefore the effective temperature is:

$$T = \left(\frac{340.1\times10^{-12}}{5.670\times10^{-8}}\right)^{0.25} = 0.28\,\mathrm{K} \qquad \text{(1 point)}$$

Theory 7: ‘Libration’

As a result of libration, studied among others by Johannes Hevelius, more than half of the Moon’s surface can be observed from Earth. Assume that the observer is geocentric.

(a) Estimate ϕ_B, the maximum angle of libration in latitude. The axial tilt (obliquity) of the Moon with respect to its orbital plane is $\alpha = 6°41'$.

(b) Estimate ϕ_L, the maximum angle of libration in longitude. Assume that the Moon is always aligned with the same side facing towards the second focus F2 of its orbit, and that the eccentricity of the Moon’s orbit e changes between 0.044 and 0.064 on a timescale of several months.

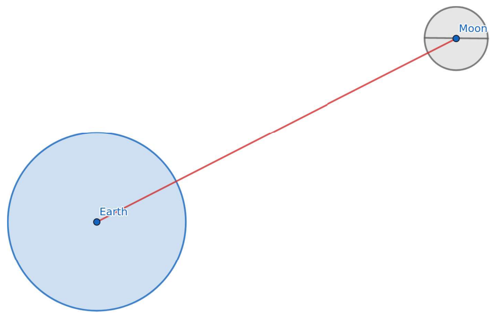
(c) Estimate the fraction of the Moon’s surface which can be seen from Earth.

(d) Calculate how many months (lunations) are needed for an observer to see the Moon’s surface determined in part (c).

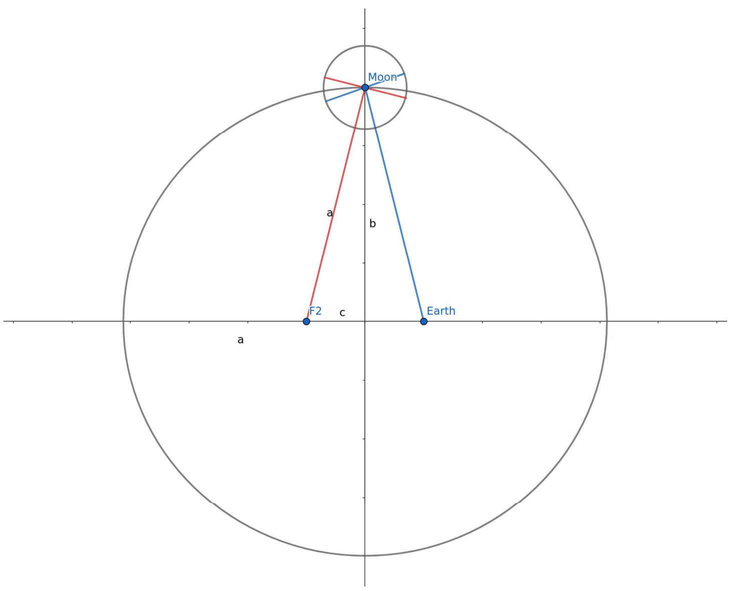
(20 points)

Solution

(a) $\phi_B = \alpha = 6°41'$ (*conceptually difficult:* 5 points)



(b) The maximum angle of libration ϕ_L occurs when the Moon is at point M and for $e = 0.064$:



ϕ_L is the Earth–Moon–F2 angle; $|EF2| = 2c$.
Therefore $\sin(\phi_L/2) = c/a = e \implies \phi_L = 2\arcsin(0.064) = 7.34° = 7°20'$ (5 points)

(c) In the less accurate approximation,

$$(2\phi_B + 2\phi_L)/180° = x/50\% \implies x = 8\%$$

and the total area visible is $S \approx 50\% + x = 58\%$.
This is less accurate as it counts the overlapping areas made visible by both librations twice.

A slightly more accurate approximation is given by taking the average of ϕ_B and ϕ_L, $7°01'$. The belt made visible by libration will thus be $2\pi R_{\text{Moon}} \cdot 7°/360° \approx 213$ km wide. The surface area of the belt is then approximately 213 km $\times\, 2\pi R_{\text{Moon}} = 2.32 \times 10^6\,\text{km}^2$, which accounts for about 6% of the lunar surface. If the students does this algebraically before substituting, they also get:

$$\frac{2\pi R_{\text{Moon}} \cdot 2\pi R_{\text{Moon}}}{4\pi 2 R_{\text{Moon}}^2} \cdot \frac{7°}{360°} = \pi \frac{7°}{360°} \approx 6\%.$$

In either case the total area visible is $S \approx 50\% + 6\% = 56\%$.

(less accurate solution 4 points, better solution 5 points)

(d) It is necessary to find a common multiple of the period of libration in latitude $P_B = 27.2122$ d (the draconic month) and the period of libration in longitude $P_L = 27.5545$ d (the anomalistic month).

The simplest way is analogous to synodic periods:

$$\frac{1}{T_{\text{syn}}} = \frac{1}{T_1} - \frac{1}{T_2}$$

thus

$$\frac{1}{T} = \frac{1}{P_B} - \frac{1}{P_L} \implies T \approx 2190.5\,\text{d} = 74\,\text{lunations}.$$

Or using continued fractions:

$$\frac{P_L}{P_B} = \frac{275545}{272122} \approx 1 + \frac{1}{79 + \frac{1}{2 + \frac{1}{131 + \frac{1}{6 + \ldots}}}}.$$

With the first term we approximate $P_L/P_B \approx 80/79$; since $80 \times 27.2122 \approx 2176.9$ d and $79 \times 27.5545 \approx 2176.8$ d this is already close enough, and T is again equal to about 74 lunations.

(3 points)

Since we only need half the cycle to see everything once (as the Moon swings away and back again during the whole cycle) therefore $T/2 = 37$ months ≈ 3 years is enough to see all of the potentially visible parts of the Moon's surface.

(2 points)

Theory 8: 'Neutrinos'

In a simplified model of a supernova explosion, the core of a star, composed of pure iron $^{56}_{26}\mathrm{Fe}$ nuclei with a total mass of 1 $M_\odot$, changes into a neutron star composed of individual electrons, protons and neutrons in numerical proportions of 1:1:8. This process is called 'neutronization' and results in the emission of a large number of neutrinos.

Calculate the solar neutrino flux on Earth. How much larger would the flux of neutrinos reaching the Earth from the supernova be than the steady neutrino emission of the Sun, if the supernova exploded in the centre of the Galaxy and the process of neutronization of the core took about 0.01 s? Give an order-of-magnitude answer.

(20 points)

Solution

Neutrino emission of the supernova:

From the table of constants, the core mass $M = 1M_\odot = 1.988 \cdot 10^{30}$ kg.
Mass of one atom of iron $^{56}_{26}\mathrm{Fe}$:

$$m_{\mathrm{Fe}} = 56\,\mathrm{Da} = 56 \times 1.661 \times 10^{-27}\,\mathrm{kg} = 9.3016 \times 10^{-26}\,\mathrm{kg}.$$

Note: atomic mass is expressed in *atomic mass units* symbol a.m.u. or u, also called *Daltons*, symbol Da. Students may use any of these notations in their solutions.

The number of $^{56}_{26}\mathrm{Fe}$ nuclei in the core is therefore:

$$n_{\mathrm{Fe}} = M/m_{\mathrm{Fe}} = 2.1373 \times 10^{55} \qquad \text{(1 points)}$$

and the initial number of nucleons is:

$$n_{\mathrm{nuc}} = 56\,n_{\mathrm{Fe}} = 1.1969 \times 10^{57}. \qquad \text{(1 points)}$$

Correspondingly, the initial number of protons is $n_{\mathrm{p}} = 26\,n_{\mathrm{Fe}}$ and the initial number of neutrons is $n_{\mathrm{n}} = 30\,n_{\mathrm{Fe}}$. (2 points)

We assume all the nucleons in the original $^{56}_{26}\mathrm{Fe}$ nuclei are converted to individual nucleons in the given proportions (1 proton : 8 neutrons). Therefore after the explosion,

$$n'_{\mathrm{n}} = \frac{8}{9} n_{\mathrm{nuc}} = 1.0634 \times 10^{57}$$

neutrons and

$$n'_{\mathrm{p}} = \frac{1}{9} n_{\mathrm{nuc}} = 1.3299 \times 10^{56}$$

protons remain, and thus

$$n_{\mathrm{p}} - n'_{\mathrm{p}} = 26\,n_{\mathrm{Fe}} - n'_{\mathrm{p}} = 4.2271 \times 10^{56} \qquad \text{(4 points)}$$

protons are changed. Since 1 neutrino is produced for every converted proton ($\mathrm{p} + \mathrm{e}^- \to \mathrm{n} + \nu_{\mathrm{e}}$), the neutrino flux is:

$$I_\nu = \frac{4.2271 \times 10^{56}}{0.01\,\mathrm{s}} = 4.2271 \times 10^{58}\,\text{neutrinos s}^{-1}. \qquad \text{(2 points)}$$

The flux density observed at Earth is given by:

$$F_{\rm SN} = \frac{I_\nu}{4\pi d_{\rm Gal}^2}.$$

Substituting $d_{\rm Gal} = 8$ kpc $= 8000 \times 3.086 \times 10^{16}\,\rm m = 2.4688 \times 10^{20}\,\rm m$,

$$F_{\rm SN} \approx 5.5 \times 10^{16}\,\text{neutrinos s}^{-1}\,\text{m}^{-2}. \qquad \text{(2 points)}$$

Neutrino flux from the Sun:

We can assume that the primary source of Solar luminosity is the p–p reaction,

$$4\rm p^+ + 2e^- \rightarrow ^4 He^{2+} + 2\nu_e.$$

Neglecting the electrons and neutrinos, the change in mass Δm is:

$$\Delta m = 4 \cdot 1.673 \times 10^{-27}\,\rm kg - 6.645 \times 10^{-27}\,kg = 4.7 \times 10^{-29}\,kg,$$

and thus from $E = (\Delta m)c^2$ the amount of energy released is $\approx 4.2244 \times 10^{-12}$ J.
Solar luminosity is $L_\odot = 3.826 \times 10^{26}$ W, so the number of reactions per second is approximately:

$$\frac{3.826 \times 10^{26}}{4.2244 \times 10^{-12}} = 9.057 \times 10^{37}.$$

With 2 neutrinos released per reaction, this gives a Solar neutrino flux of:

$$I_{\nu\odot} \approx 1.8 \times 10^{38}\,\text{neutrinos s}^{-1}.$$

(If the student remembers that the p–p reaction releases 26.73 MeV $=$ 4.2826×10^{-12} J, they will get 8.9338×10^{37} reactions per second and the same final answer.)

(getting to $I_{\nu\odot}$: 5 points)

At the distance of Earth, the observed flux density is given by:

$$F_\odot = \frac{I_{\nu\odot}}{4\pi d_\odot^2},$$

where $d_\odot = 1$ au $= 1.496 \times 10^{11}$ m, and so:

$$F_\odot \approx 6.4 \times 10^{14}\,\text{neutrinos s}^{-1}\,\text{m}^{-2}. \qquad \text{(1 point)}$$

The final ratio is therefore:

$$F_{SN}/F_\odot \approx \frac{5.5 \times 10^{16}}{6.4 \times 10^{14}} \approx 86 \approx 100,$$

or two orders of magnitude. (2 points)

Theory 9: 'Second eclipse'

For each of two eclipsing binary systems, Bolek and Lolek, the primary eclipses were observed with very high cadence as depicted below:

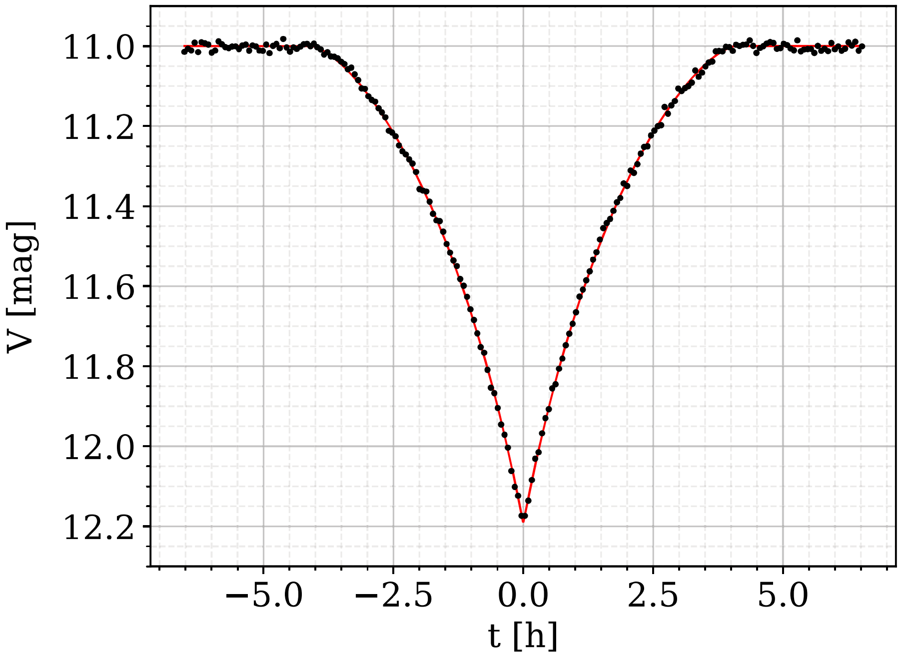


Figure 1: Observed lightcurve for system Bolek.

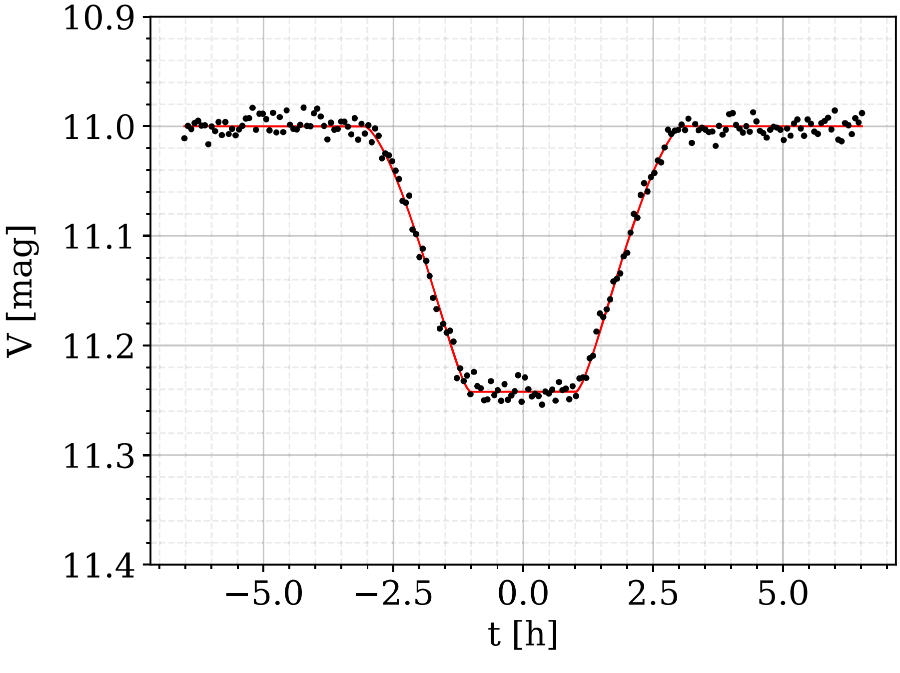


Figure 2: Observed lightcurve for system Lolek.

In the figures, t is the time in hours relative to the moment of minimum and V is the brightness in the V (visible) band in magnitudes. The points are the measurements and the line is the fitted model of the shape of the eclipse.

You can assume that in both cases the eclipses are central ($i = 90°$) and last for a very small fraction of the orbital period, limb darkening is negligible, and the orbits have low eccentricity.

On the Answer Sheet, draw the predicted shape of the light curve for each of the secondary eclipses. Write down the equations and calculations leading to your predictions.

(20 points)

Answer Sheet

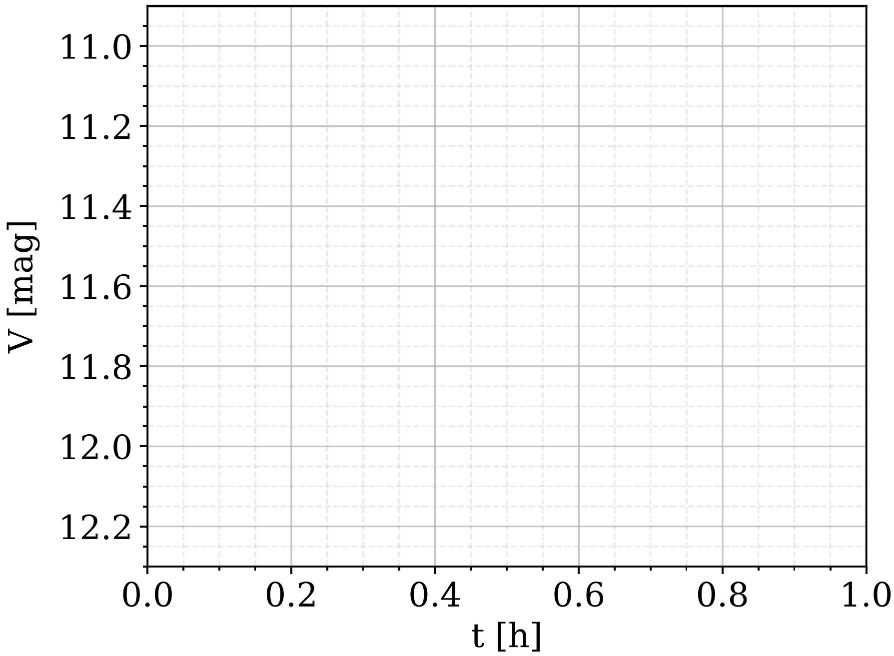


Figure 3: Predicted lightcurve for the second eclipse for system Bolek.

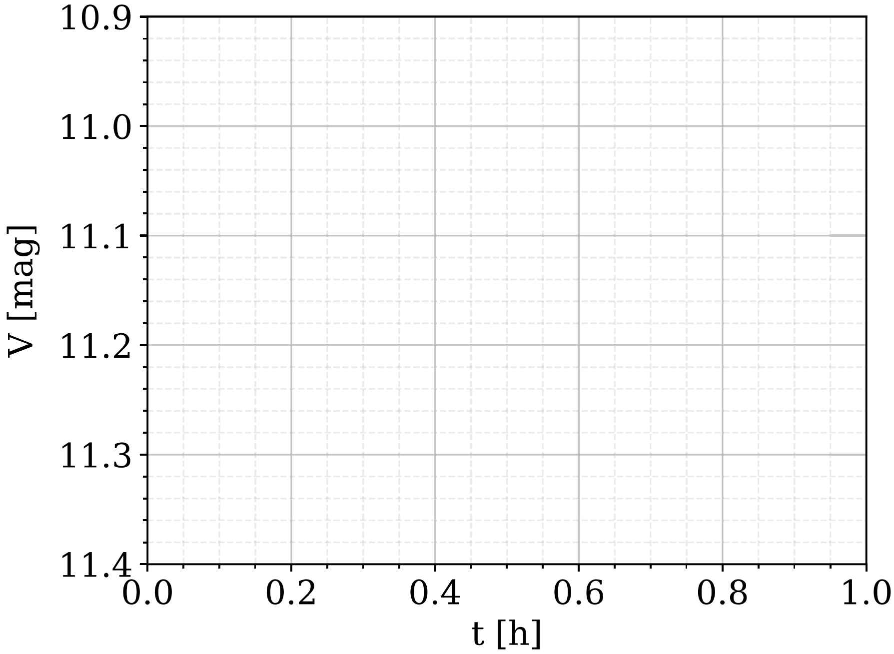


Figure 4: Predicted lightcurve for the second eclipse for system Lolek.

Solution

For each of the systems, since the eclipses are short we can assume that the angle swept along the orbit during the eclipse is negligibly small, which means that the tangential velocity in relative orbital motion is constant.

System A

A sharp peak minimum in a central eclipse implies that $R_1 = R_2$. Therefore, both eclipses will be total, with minima of brightnesses m_1 and m_2, respectively.

Noticing that $R_1 = R_2$	1 point
Brightnesses at the two minima are m_1 and m_2	1 point

Define the magnitude at minimum as $m_{\text{min}} = m_1$ and the baseline magnitude as $m_{1,2} = m_1 + m_2$. From Pogson's law applied to the two cases: $m_1 - m_{1,2} = -2.5 \log_{10} \left(L_1/(L_1 + L_2)\right)$ and $m_2 - m_1 = -2.5 \log_{10} \left(L_2/L_1\right)$. For the first case, with simple operations we obtain: $L_2/L_1 = 10^{(m_1 - m_{1,2})/2.5} - 1$. Plugging this into the second case: $m_2 = m_1 - 2.5 \log_{10} \left(10^{(m_1 - m_{1,2})/2.5} - 1\right)$.

Correct application of Pogson's law	1 point
Obtaining the luminosity ratio	1 point
Final formula for the depth of the minimum	1 point

As the orbits are near-circular, timescales will be the same for both eclipses. Therefore it is sufficient to 'copy' the shape of the primary one while scaling down the depth to $|m_{1,2} - m_2|$. Reading off $m_{1,2}$ and m_{min} from the plot, we arrive at $L_2/L_1 = 2$, $m_2 = 11.44$, and the following figure:

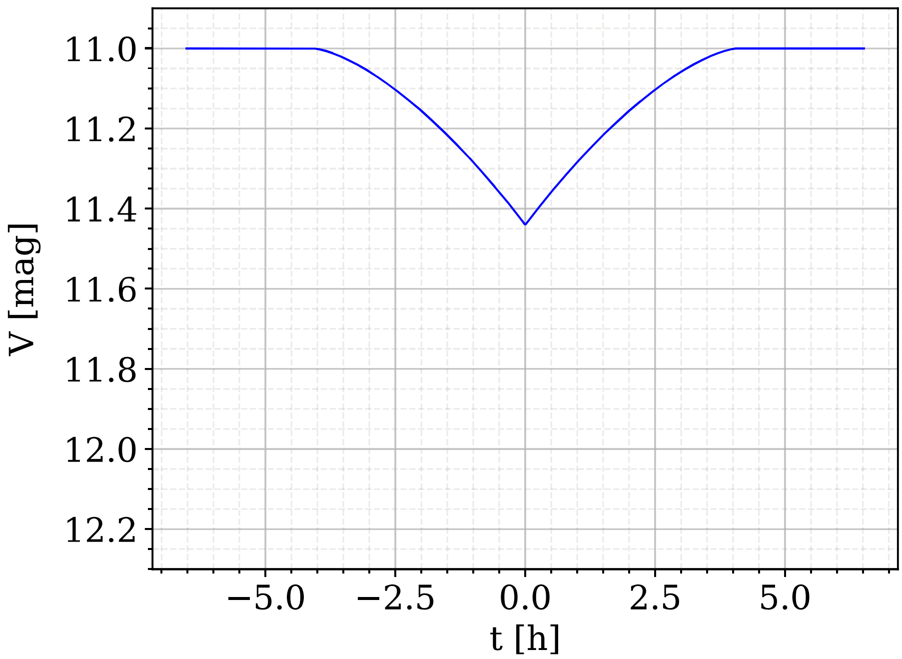


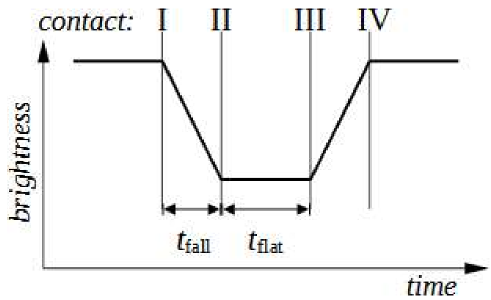
Figure 5: Predicted lightcurve for the second eclipse for system A – correct answer.

Correct depth of the second eclipse	2 points
Overall correct shape (sharp peak and symmetry not broken)	1 point
All contact times matching the first eclipse	1 point

Note about marking: Reproducing the exact shape of the eclipse (concave/convex, smooth descent, etc.) is not subject to grading as it requires mm-level precision, and the focus is on the thought process instead.

System B

For the second system, we follow a generally similar scheme, but we see that the minimum is flat. t_{fall} is the time of the descent from the baseline brightness to the lowest point, i.e., between contacts I and II, while t_{flat} is the time between contacts II and III during which the lightcurve remains at its lowest point:



From this, we obtain $t_{\text{flat}}/t_{\text{fall}} = 2(R_2 - R_1)/2R_1$ and, after rearranging, $R_2/R_1 = t_{\text{flat}}/t_{\text{fall}} + 1$.

Noticing that $R_1 \neq R_2$ and R_2/R_1 can be calculated from the contact times	1 point
Calculating, or giving the correct formula for, R_2/R_1	1 point

Here R_1 is the radius of the smaller star and there are two solutions – it could either be (partially) *eclipsing* or (entirely) *eclipsed*. Let us first assume that it was eclipsed. After measuring m_{min} for the eclipse and $m_{1,2}$ for the baseline we again (just like for system A) assume $m_{\text{min}} = m_1$ and write down: $L_1/L_2 = 10^{(m_1 - m_{1,2})/2.5} - 1$.

Calculating, or giving the correct formula for, L_2/L_1	1 point

Reading off $m_{1,2}$ and m_{min} from the plot, we arrive at $L_2/L_1 = 4$, $R_2/R_1 = 2$.
$L_i = S_i \pi R_i^2$, where S_i stands for the surface brightness of each star.
$\implies S_1 = S_2 = S$ (or $T_{\text{eff},1} = T_{\text{eff},2} = T_{\text{eff}}$).

Relation between L, R ratios and S (or T_{eff}) ratios	1 point
Concluding that $S_1 = S_2$ (or $T_{\text{eff},1} = T_{\text{eff},2}$)	2 points

In such a case, *it does not matter* if star 2 is eclipsing or eclipsed. During the baseline, we always see two disks with a total brightness of $S(\pi R_1^2 + \pi R_2^2)$, while during the eclipse, we see one disk of radius R_2 and total brightness $S\pi R_2^2$.

Note about marking: This solution can be independently verified by assuming that the smaller component was eclipsing and cutting out a disk of surface πR_1^2 and brightness contribution $S_1 \pi R_1^2$ in the star of surface πR_2^2 and surface brightness S_2. Instead of $m_{\text{min}} = m_1$ of star 1, we would have measured $m_{\min} = m_{\mathrm{ecl},1}$ of a combination of the two, with a brightness $L_{ecl1} = S_1\pi R_1^2 + S_2\pi(R_2^2 - R_1^2)$ The same conclusion $S_1 = S_2 = S$ will follow. Any solution correctly arriving at this conclusion and the correct figure – no matter the order of steps taken – should be scored equally. For the maximum number of points, however, the student should make a convincing point that this is the *only* solution, i.e. lifting the previously made assumption (or exploring both cases).

Solution showing there is only one possible shape of the secondary eclipse	1 point

To draw the second eclipse, one must simply copy the shape of the first one:

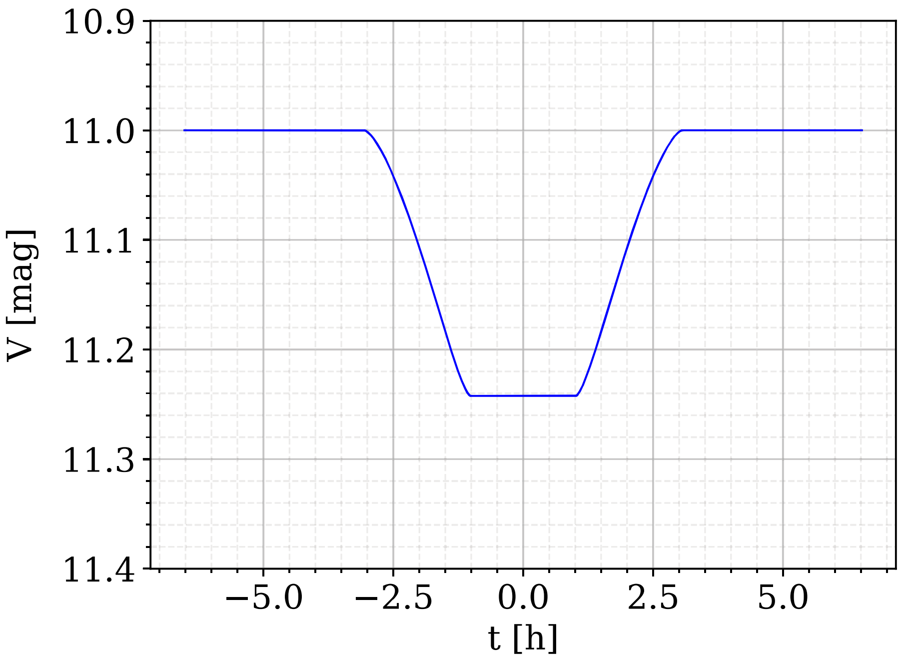


Figure 6: Predicted lightcurve for the second eclipse for system B – correct answer.

Correct depth of the second eclipse	2 points
Overall correct shape (flat minimum and symmetry not broken)	1 point
All contact times matching the first eclipse	1 point

Theory 10: 'Aldebaran'

On 9 March 1497, Nicolaus Copernicus observed the occultation of Aldebaran by the Moon from Bologna. In his work *De revolutionibus orbium cœlestium*[1] Copernicus described the event: "I saw the star touching the dark edge of the Moon and disappearing at the end of the 5th hour of the night between the horns of the Moon, closer to the south horn by a third of the Moon's diameter."

Assuming that the occultation was observed on the local meridian, that at maximum occultation Aldebaran was $0.32'$ above the southern edge of the Moon, and that the apparent angular diameter of the Moon as seen from Bologna was $31.5'$, solve the following tasks:

(a) Find the latitude φ_1 of a place with the same longitude as Bologna, from which Aldebaran would have appeared to pass behind the centre of the Moon.

(b) Find the duration of the occultation as seen from latitude φ_1 if Aldebaran appeared to pass along the diameter of the lunar disk. For simplicity, also assume that the Moon and the observer are moving linearly at constant speed, that the Moon's orbit is circular and that the declination of the Moon does not change during the occultation.

(c) Find the topocentric angular velocity of the Moon against the background stars during the occultation for an observer at latitude φ_1, in arcmin/hour, applying the same assumptions as in part (b).

(d) Estimate the range of the Moon's topocentric angular velocities (against the background stars) in arcmin/hour at latitude φ_1, assuming a circular orbit. Show how this result can be justified by expressing the relative velocity of the Moon and observer in terms of their velocity vectors.

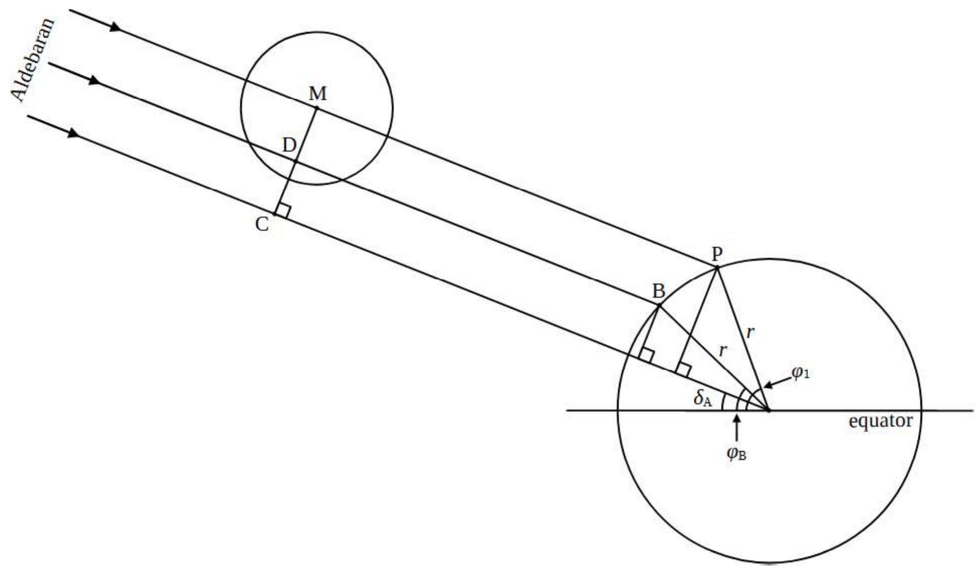
The declination of Aldebaran was $\delta_{\mathrm{A}} = 15.37°$ in 1497 (due to precession), and the latitude of Bologna is $\varphi_{\mathrm{B}} = 44.44°$ N.

(25 points)

[1] Book VI, Chapter 27

Solution

(a) Aldebaran is far enough away that the light rays from it can be considered to be parallel, thus the problem can be modelled by the following diagram:



where the light ray through C passes through the centre of the Earth and thus δ_A is the declination of Aldebaran; the ray through D and B (Bologna) represents the situation observed by Copernicus, with Aldebaran behind the Moon and $0.32'$ above the south edge; and the ray through M and P represents the situation where Aldebaran is exactly behind the centre of the Moon. φ_B is the geocentric latitude of Bologna, φ_1 is the latitude of the place P and r is the radius of the Earth (assume the Earth is spherical).

Calculating the latitude φ_1 is thus a matter of simple geometry, where:

$$|CD| = r\sin(\varphi_B - \delta_A),$$

$$|CM| = r\sin(\varphi_1 - \delta_A),$$

and the distance $|DM|$ is the fraction of the Moon's radius given by:

$$|DM| = 1\,737\,\text{km} \times \left(1 - \frac{0.32'}{31.5'/2}\right) = 1\,702\,\text{km}.$$

Therefore, since

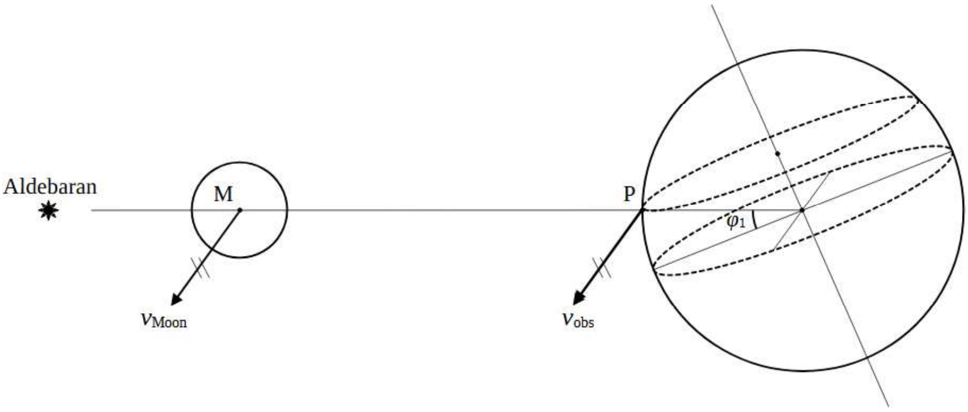
$$|CM| = |CD| + |DM| = 3\,099\,\text{km} + 1\,702\,\text{km} = 4\,801\,\text{km},$$

$$\implies \varphi_1 = \arcsin\left(\frac{4\,801}{6\,378}\right) + \delta_A = 64.19°. \qquad \text{(6 points)}$$

(b) Given the assumptions, the duration of the occultation will just depend on the difference between the tangential speeds of the observer, v_{obs}, and Moon, v_{Moon}, and on the diameter of the Moon.

For the observer, the tangential speed is given by the circumference of their path at latitude φ_1 divided by the sidereal day:

$$v_{\text{obs}} = \frac{2\pi r_{\text{Earth}} \cos\varphi_1}{23^{\text{h}}\,56\text{m}\,4^{\text{s}}} = 729\,\text{km/h}$$



Similarly for the Moon, using the semi-major axis of the Moon's orbit and the length of the sidereal month (from the table of constants):

$$v_{\text{Moon}} = \frac{2\pi \times 3.844 \times 10^5\,\text{km}}{27.321661\,\text{d} \times 24} = 3\,683\,\text{km/h}$$

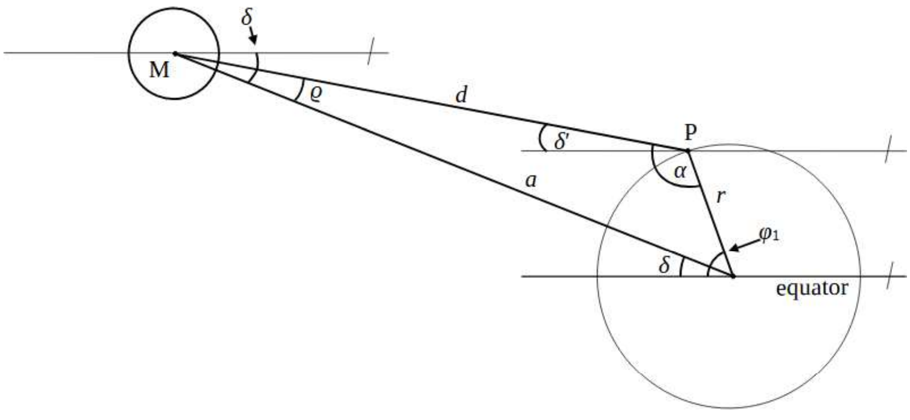
The difference of speeds is then:

$$v_{\text{rel}} = v_{\text{Moon}} - v_{\text{obs}} = 2\,954\,\text{km/h}$$

and the time taken to move across the lunar diameter ($2 \times 1737\,\text{km}$) at that speed is:

$$t = \frac{2 \times 1\,737\,\text{km}}{2\,954\,\text{km/h}} = 1.176 \approx 1.18\,\text{hours.} \qquad \text{(5 points)}$$

(c) To determine the topocentric angular velocity, we need to find the distance d between the observer (at place P at latitude φ_1) and the Moon:



where a is the semi-major axis of the Moon's orbit, δ is the geocentric declination of the Moon, and δ' is the topocentric declination of the Moon as seen by the observer. We can rewrite $\varrho = \delta - \delta'$ and $\alpha = 180^\circ - (\varphi_1 - \delta) - \varrho = 180^\circ - (\varphi_1 - \delta) - (\delta - \delta') = 180^\circ - (\varphi_1 - \delta')$, then using the sine theorem:

$$\frac{d}{\sin(\varphi_1 - \delta)} = \frac{a}{\sin\left(180^\circ - (\varphi_1 - \delta')\right)} = \frac{r}{\sin(\delta - \delta')}$$

Simplifying and substituting $\delta' = \delta_\mathrm{A}$:

$$\frac{d}{\sin(\varphi_1 - \delta)} = \frac{a}{\sin(\varphi_1 - \delta_\mathrm{A})} = \frac{r}{\sin(\delta - \delta_\mathrm{A})}$$

We can now calculate δ:

$$\delta = \arcsin\left(\frac{r\sin(\varphi_1 - \delta_\mathrm{A})}{a}\right) + \delta_\mathrm{A} = 16.08^\circ$$

and d:

$$d = a\frac{\sin(\varphi_1 - \delta)}{\sin(\varphi_1 - \delta_\mathrm{A})} = 380\,203\,\mathrm{km}.$$

Thus the angular velocity is:

$$\omega = \left(\frac{2\,R_\mathrm{Moon}}{d}\right)\left(\frac{1}{1.176\,\mathrm{h}}\right)\mathrm{rad/h} = \left(\frac{2\times 1\,737}{380\,203}\right)\left(\frac{1}{1.176\,\mathrm{h}}\right)\left(\frac{180\times 60}{\pi}\right) = 26.71 \approx 27\,\mathrm{arcmin/h}.$$

(7 points)

(d) The result in part (c), 27 arcmin/h, was calculated for the situation when the Moon and observer are moving in the same direction, reducing their relative velocity, and thus represents the lower limit of the range of topocentric angular velocities. The upper limit is given by the situation when the Moon and observer are moving in the opposite direction, i.e. when the Moon is due north.

Using the calculation from part (b), we find that the relative linear speed is:

$$v'_\mathrm{rel} = v_\mathrm{Moon} + v_\mathrm{obs} = 3\,683 + 729 = 4\,412\,\mathrm{km/h}$$

Neglecting the effect of the slightly greater distance between the observer and the Moon when the Moon is on the northern side of the sky, the angular speed will be greater by the ratio of the linear speeds,

$$\omega' = \omega\,\frac{4\,416}{2\,954} = 39.89 \approx 40\,\mathrm{arcmin/h},$$

thus the range is approximately 27–40 arcmin/h. (3 points)

If the additional distance is taken into account, the upper limit will be around 1% less.

Expressing the velocities as vectors,

$$\vec{v}_\mathrm{rel} = \vec{v}_\mathrm{Moon} - \vec{v}_\mathrm{obs},$$

the magnitude of the relative velocity, $v_\mathrm{rel} = |\vec{v}_\mathrm{rel}|$ is given by the scalar projection of $\vec{v}_\mathrm{obs}$ on $\vec{v}_\mathrm{Moon}$:

$$v_\mathrm{rel} = v_\mathrm{Moon} - v_\mathrm{obs}\cos\alpha$$

where α is the angle between the vectors. From this it can be seen that the relative velocity must be minimum for $\alpha = 0^\circ$ and maximum for $\alpha = 180^\circ$. These conditions are met on the local meridian, due south or due north, respectively.

(4 points)

Theory 11: 'X-ray emission from galaxy clusters'

Clusters of galaxies are strong X-ray sources. It has been established that the emission mechanism is thermal bremsstrahlung (free-free radiation) from a hot hydrogen and helium plasma inside the cluster. The luminosity L_X (in Watts) of each component of the plasma is described by the formula:

$$L_X \;=\; 6\times 10^{-41}\, N_e\, N_X\, T^{\frac{1}{2}}\, V\, Z_X^2,$$

where the symbols represent:

$X-$ Hydrogen (H) or Helium (He),
N_e- number density of electrons [m^{-3}],
N_X- number density of ions X [m^{-3}],
Z_X- atomic number of ion X,
$T-$ temperature of the plasma [K],
$V-$ volume occupied by the plasma [m^3].

(a) Determine the total mass (in solar masses) of the plasma which emits the X-rays, assuming that:

- the plasma is fully ionized with 1 helium ion for every 10 hydrogen ions;
- $L_{\text{total}} \;=\; 1.0\times 10^{37}\,\mathrm{W}$,
- $T \;=\; 80\times 10^{6}\,\mathrm{K}$,
- the plasma is uniformly distributed in a sphere of radius $R \;=\; 500\,\mathrm{kpc}$,
- self-absorption is negligible.

(16 points)

The photons of the cosmic microwave background (CMB) interact with plasma in a process known as inverse Compton scattering. The CMB normally has a thermal blackbody spectrum at a temperature of 2.73 K. However, interaction with the plasma leads to distortion of the CMB spectrum (known as the Sunyaev–Zeldovich effect).

(b) Estimate the mean free path of CMB photons in the plasma, i.e. the average distance travelled by a photon before interacting with an electron. Express it in Mpc. The effective cross section for photon–electron interactions is $\sigma = 6.65\times 10^{-29}\,\mathrm{m}^2$. (5 points)

(c) Estimate the typical energy of CMB photons. (3 points)

(d) The energy of CMB photons can be increased by a factor of up to $(1+\beta)/(1-\beta)$ due to the inverse Compton scattering, where $v = \beta c$ is the velocity of electrons. Estimate the energy of scattered CMB photons. (6 points)

(Total: 30 points)

Solution

Part (a)

Concentrations of electrons, N_e, and He nuclei, $N_{\rm He}$, is related to the concentration of H nuclei, $N_{\rm H}$:

$$N_{\rm He} \;=\; 0.1\,N_{\rm H} \qquad \text{(1 point)}$$

$$N_e \;=\; N_{\rm H} \,+\, 2\,N_{\rm He} \;=\; 1.2\,N_{\rm H} \qquad \text{(2 points)}$$

The total X-ray emission is a sum of the bremmstrahlung generated by interaction of electrons with H and He nuclei:

$$L \;=\; 6\cdot 10^{-41}\,N_e\,T^{\frac{1}{2}}\,V\,(N_{\rm H} \,+\, Z_{\rm He}^2\,N_{\rm He}),$$

where $Z_{\rm He} = 2$. The luminosity L is expressed by the concentration of H nuclei:

$$L \;=\; 6\cdot 10^{-41}\,1.2\,N_{\rm H}\,T^{\frac{1}{2}}\,V\,1.4\,N_{\rm H},$$

$$\Longrightarrow\; L = (10.08\times 10^{-41})\;T^{\frac{1}{2}}\;V\,N_{\rm H}^2 \approx\; 10^{-40}\,T^{\frac{1}{2}}\;V\,N_{\rm H}^2. \qquad \text{(3 points)}$$

The volume V is given by:

$$V \;=\; \frac{4}{3}\,\pi\,R^3 \;=\; 1.54\cdot 10^{67}\;{\rm m}^3. \qquad \text{(3 points)}$$

Thus, the concentration of $N_{\rm H}$:

$$N_{\rm H} \;=\; \left(\frac{L}{10^{-40}\,T^{\frac{1}{2}}\,V}\right)^{1/2} \;\approx\; 8.48\;\times\;10^2\;{\rm m}^{-3}. \qquad \text{(3 points)}$$

To obtain the total mass of the plasma, M, one should multiply the volume V by the the sum of H and He mass densities:

$$M \;=\; V\,(N_{\rm H}\,m_{\rm H} \,+\, N_{\rm He}\,m_{\rm He}) \;=\; 3.03\times 10^{43}\,{\rm kg} \;\approx\; 1.52\;\times\;10^{13}\,{\rm M}_{\odot}. \qquad \text{(4 points)}$$

where $m_{\rm H}$ and $m_{\rm He}$ are the masses of hydrogen and helium atoms.

Part (b)

Let L be the mean free path of a photon. The number of electrons in a cylinder with a cross section area of σ and a length of L equals

$$N = nL\sigma = 1 \;\Longrightarrow\; L = 1/(n\sigma).$$

Assuming $n = N_e = 1.2N_{\rm H} = 1.018\times 10^3\,{\rm m}^{-3}$, we get

$$L = \frac{1}{1.018\times 10^3\cdot 6.65\times 10^{-29}} = 1.48\times 10^{25}\,{\rm m} = 4.79\times 10^8\,{\rm pc} \approx 500\,{\rm Mpc}. \qquad \text{(5 points)}$$

(The radius of the cluster is $\sim$ 1000 times smaller than L, so the interactions between the CMB photons and hot electrons are rare.)

Part (c)

Using Wien's law,

$$\lambda = \frac{b}{T} = \frac{2.898\times 10^{-3}}{2.73} = 1.07\times 10^{-3}\;{\rm m}$$

$$\Longrightarrow\; E_0 = \frac{hc}{\lambda} = 1.85\times 10^{-22}\;{\rm J} \qquad \text{(4 points)}$$

Part (d)

We first have to estimate the typical velocity v_e of electrons using the formula for the kinetic energy of the particles in a gas.

$$\frac{1}{2}mv^2 = \frac{3}{2}kT \implies v_e = \sqrt{\frac{3kT}{m_e}}$$

$$\therefore v_e = \sqrt{\frac{3 \cdot 1.381 \times 10^{-23} \cdot 8 \times 10^7}{9.109 \times 10^{-31}}} = 6.032 \times 10^7 \ \mathrm{m\,s^{-1}} = 0.202c$$

(3 points)

The energy of upscattered photons is:

$$E' = \frac{1+\beta}{1-\beta}E_0 \approx 1.5E_0 = 2.78 \times 10^{-22} \text{ J.} \qquad \text{(2 points)}$$

Note: The formula for the kinetic energy of particles in an ideal gas remains valid even at such high temperatures. This is because

$$x = \frac{m_e c^2}{kT} = \frac{9.109 \times 10^{-31} \cdot (2.998 \times 10^8)^2}{1.381 \times 10^{-23} \cdot 8 \times 10^7} = 74.1 \gg 1,$$

and the electrons can be treated as non-relativistic. The full relativistic formula for the rms velocity of particles in an ideal gas is

$$< v_e^2 >= \frac{xc^2}{K_2(x)} \int_0^\infty \frac{\sinh^4 \phi}{\cosh \phi} e^{-x \cosh \phi} d\phi,$$

where $K_2(x)$ is a modified Bessel function of the second kind. It can be shown that for $x = 74.1$ this formula yields $\sqrt{< v_e^2 >} = 0.198c$, which is virtually identical to the prediction of the non-relativistic formula.

Theory 12: 'DART'

The Double Asteroid Redirection Test (DART) was a NASA mission to evaluate a method of planetary defense against near-Earth objects. The spacecraft hit Dimorphos, a moon of the asteroid Didymos, to study how the impact affected its orbit.

(a) Calculate the expected orbital period change (in minutes), assuming that the collision was head-on, central, and perfectly inelastic.

Assume that before the impact Dimorphos orbited Didymos on a circular orbit with a period of $P = 11.92$ h. The masses of Dimorphos and Didymos are $m = 4.3 \times 10^9\,\mathrm{kg}$ and $M = 5.6 \times 10^{11}\,\mathrm{kg}$, respectively. The mass and speed of the DART spacecraft relative to Dimorphos at a moment of impact were $m_\mathrm{s} = 580\,\mathrm{kg}$ and $v_\mathrm{s} = 6.1\,\mathrm{km\,s^{-1}}$. Neglect the gravitational influence of other bodies.

(20 points)

(b) In reality, the orbital period of Dimorphos was observed to be changed by $\Delta P_0 = -33\,\mathrm{min}$. This is due to the momentum transfer associated with the recoil of the ejected debris: the spacecraft was absorbed by the asteroid, but the impact excavated some material from the asteroid and ejected it into space. Calculate the momentum of the ejected debris and express it as a fraction of the momentum of Dimorphos before the collision. You can assume that the mass of the ejected material is much smaller than the mass of Dimorphos.

(15 points)

(c) Calculate the velocity change (in $\mathrm{mm\,s^{-1}}$) of Dimorphos as a result of the impact, taking into account the effect of the ejected debris.

(5 points)

(Total: 40 points)

Solution

Part (a)

Didymos' mass is much larger than Dimorphos' mass. Therefore, the radius of the orbit of Dimorphos before impact, a, can be calculated from Kepler's 3rd law:

$$\frac{GM}{4\pi^2} = \frac{a^3}{P^2},$$

and therefore:

$$a = \left(\frac{GMP^2}{4\pi^2}\right)^{1/3} = 1.2\,\mathrm{km}.$$

The orbital velocity of Dimorphos before impact was:

$$v_0 = \frac{2\pi a}{P} = 0.176\,\mathrm{m/s}. \qquad \text{(2 points)}$$

Let v' be the Dimorphos velocity right after the collision. Using the law of conservation of momentum, we have:

$$mv_0 - m_s v_s = (m + m_s)v',$$

and:

$$v' = \frac{mv_0 - m_s v_s}{m + m_s} \approx v_0 - \frac{m_s}{m} v_s,$$

where we used the fact that the mass of the spacecraft mass is much smaller than the mass of Dimorphos.
We then use the vis-viva equation to calculate the semi-major axis of the orbit after collision a':

$$(v')^2 = GM\left(\frac{2}{a} - \frac{1}{a'}\right) = \frac{GM}{a}\left(2 - \frac{a}{a'}\right) = v_0^2\left(2 - \frac{a}{a'}\right),$$

so

$$2 - \frac{a}{a'} = \left(\frac{v'}{v_0}\right)^2 = \left(1 - \frac{m_s}{m}\frac{v_s}{v_0}\right)^2 \approx 1 - \frac{2m_s}{m}\frac{v_s}{v_0}.$$

Thus, the semi-major axis changed by:

$$\frac{\Delta a}{a} = \frac{a' - a}{a} = -\frac{2m_s}{m}\frac{v_s}{v_0}. \qquad \text{(8 points)}$$

If the semi-major axis changes from a to $a + \Delta a$, then the orbital period changes from P to $P + \Delta P$, and the mass of the spacecraft can be neglected. Then:

$$\frac{a^3}{P^2} = \frac{(a + \Delta a)^3}{(P + \Delta P)^2} = \frac{a^3(1 + \Delta a/a)^3}{P^2(1 + \Delta P/P)^2} = \frac{a^3}{P^2}\left(1 + \frac{3\Delta a}{a} - \frac{2\Delta P}{P}\right),$$

hence:

$$\frac{\Delta P}{P} = \frac{3}{2}\frac{\Delta a}{a}.$$

Thus, the orbital period changes by:

$$\frac{\Delta P}{P} = \frac{3}{2}\frac{\Delta a}{a} = -\frac{3m_s}{m}\frac{v_s}{v_0} \qquad \text{(8 points)}$$

We therefore expect that the orbital period of Dimorphos should decrease by 1.4%, that is, 10 minutes. (2 points)

Alternative solution (requires better numerical precision)

$$v_0 = 0.17622 \text{ m/s}$$

$$v' = 0.17662 - 0.00082 = 0.17540 \text{ m/s}$$

$$a' = 0.99080a = 1192.5 \text{ m}$$

$$P' = 705.4 \text{ min}$$

Hence $\Delta P = 705.4 - 715.2 = -9.8 \approx -10$ min. I propose to grant full points ONLY if the first four significant figures match the solution. If the first three significant figures match the solution, grant 80% of the points. Otherwise (if the method is correct), grant HALF of the points.

Part (b)

Let Δp be the momentum of the ejected debris. Then, the momentum conservation equation becomes:

$$mv_0 - m_s v_s = (m + m_s)v' + \Delta p, \qquad \text{(4 points)}$$

so:

$$v' = \frac{mv_0 - m_s v_s - \Delta p}{m + m_s} \approx v_0 - \frac{m_s}{m} v_s - \frac{\Delta p}{m}.$$

Using similar calculations as in point a), we get:

$$\frac{\Delta a}{a} = -2\left(\frac{m_s}{m}\frac{v_s}{v_0} + \frac{\Delta p}{mv_0}\right), \qquad \text{(8 points)}$$

hence:

$$\frac{\Delta P_0}{P} = \frac{3}{2}\frac{\Delta a}{a} = -3\left(\frac{m_s}{m}\frac{v_s}{v_0} + \frac{\Delta p}{mv_0}\right).$$

Thus:

$$\frac{\Delta p}{mv_0} = -\frac{\Delta P_0}{3P} - \frac{m_s}{m}\frac{v_s}{v_0} = 0.011. \qquad \text{(3 points)}$$

Alternative solution (requires better numerical precision)

The orbital period after the collision is $P' = 11.92 - 33/60 = 11.37$ h. Thus, the semi-major axis of the orbit (after the collision) is

$$a' = \left(\frac{GMP'^2}{4\pi^2}\right)^{1/3} = 1166\,\text{m}.$$

The velocity of Dimorphos right after the collision is $v' = v_0\sqrt{2 - a/a'} = 0.1734\,\text{m}\,\text{s}^{-1}$, so $\Delta p/mv_0 = 0.011$. I propose to grant full points ONLY if the first four significant figures match the solution. If the first three significant figures match the solution, grant 80% of the points. Otherwise (if the method is correct), grant HALF of the points.

Part (c)

$$\Delta v = v' - v_0 = -\frac{m_s}{m}v_s - \frac{\Delta p}{m} = -\frac{m_s}{m}v_s + \frac{\Delta P_0}{3P}v_0 + \frac{m_s}{m}v_s = \frac{\Delta P_0}{3P}v_0 = -2.7\,\text{mm/s}.$$

(5 points)

Theory 13: 'LISA'

The Laser Interferometer Space Antenna (LISA) is a proposed experiment to detect low-frequency gravitational waves. It consists of three spacecraft arranged in an equilateral triangle. A passing gravitational wave changes the distance between the spacecraft, which can be precisely measured (more details are given in the notes below).

One of the sources of low-frequency gravitational waves are compact binary star systems, for example binary white dwarfs. Such a system was recently discovered at a distance of 2.34 kpc from the Sun. The orbital period of the binary was found to be 414.79 s and is changing at a rate of $-7.49 \times 10^{-4}\,\mathrm{s\,yr^{-1}}$ due to the emission of gravitational waves.

(a) Check if this binary system can be detected by LISA. (25 points)

(b) Calculate the chirp mass. (5 points)

(c) Determine the masses of both components knowing that the ratio between the radius of one of the components to the semi-major axis of the orbit is 0.139, and assuming both components follow the mass–radius relation for white dwarfs given in the table below. (15 points)

(Total: 45 points)

Notes:

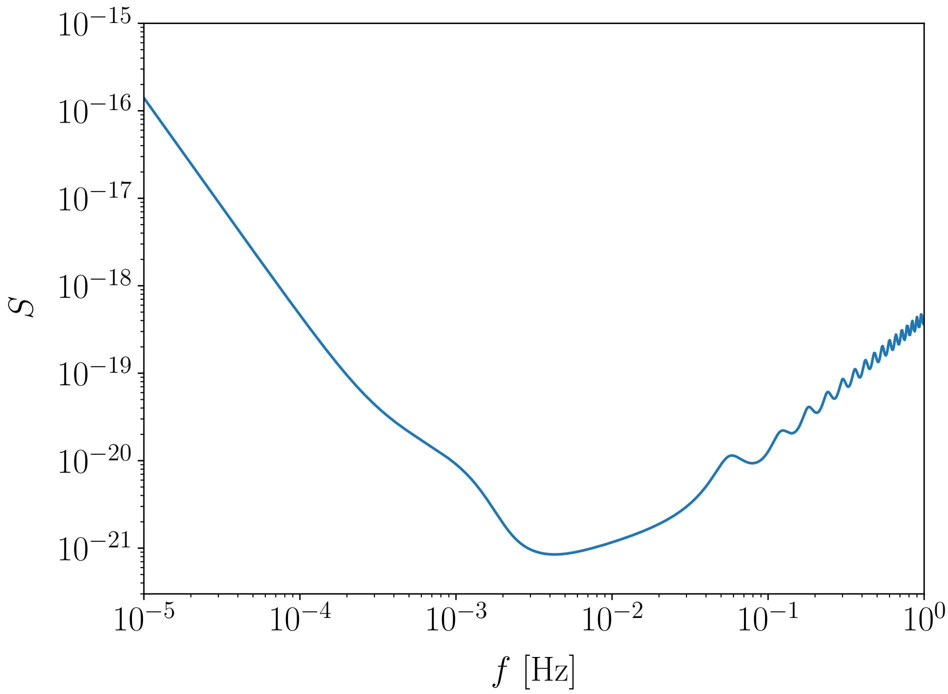
1. A binary star system with an orbital period P emits gravitational waves with a frequency of $f = 2/P$.

2. LISA measures a dimensionless quantity called the characteristic strain amplitude, S, given by
$$S = h\sqrt{fT_{\mathrm{obs}}},$$
where $T_{\mathrm{obs}} = 4\,\mathrm{yr}$ is the expected duration of the mission. h is the gravitational wave strain, given by:
$$h = \frac{2(G\mathcal{M})^{5/3}(\pi f)^{2/3}}{c^4 D},$$
where $\mathcal{M}$ is the so-called chirp mass, f is the frequency of the gravitational wave and D is the distance to the system. If we denote the masses of the components of the binary as M_1 and M_2, then the chirp mass is given by:
$$\mathcal{M} = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}}.$$
The expected sensitivity of LISA as a function of a gravitational wave frequency is presented on the figure below.

3. The semi-major axis a of the binary system changes due to the emission of gravitational waves at a rate:
$$\frac{\Delta a}{\Delta t} = -\frac{64}{5}\frac{G^3}{c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^3}.$$

M ($M_\odot$)	R ($R_\odot$)
0.48	0.0144
0.50	0.0147
0.52	0.0150
0.54	0.0153
0.56	0.0156
0.58	0.0159
0.60	0.0162
0.62	0.0165
0.64	0.0168

Mass–radius relation for white dwarfs based on theoretical models of Althaus et al. (2013) for white dwarfs of $log_g = 7.7$.



The expected sensitivity of LISA as a function of gravitational wave frequency.

Solution

Part (a)

To determine whether the system can be detected by LISA, we need to determine two quantities: the gravitational-wave frequency and the characteristic strain amplitude.

It is straightforward to calculate the gravitational-wave frequency:

$$f = \frac{2}{P} = \frac{2}{414.79} = 4.8 \times 10^{-3}\,\text{Hz}.$$

This frequency is within the LISA band and close to the maximum LISA sensitivity.

Calculating gravitational wave frequency	2 points

To estimate the characteristic strain amplitude, we need to know the chirp mass $\mathcal{M}$. The other required quantities (such as the gravitational wave frequency, distance, and duration of the LISA observations) are already known.

The orbital period change rate $\Delta P/\Delta t$ is given in the problem. We need to link it to $\Delta a/\Delta t$ which we know from Note 3 is linked to the mass function. From Kepler's third law, we know that:

$$\frac{a^3}{P^2} = \frac{G(M_1 + M_2)}{4\pi^2},$$

where M_1 and M_2 are masses of both components of the system. If, due to the emission of gravitational waves, the semi-major axis changes from a to $a + \Delta a$, then the orbital period changes from P to $P + \Delta P$, as the masses of both components are constant. Then:

$$\frac{a^3}{P^2} = \frac{(a+\Delta a)^3}{(P+\Delta P)^2} = \frac{a^3(1+\Delta a/a)^3}{P^2(1+\Delta P/P)^2} = \frac{a^3}{P^2}\left(1 + \frac{3\Delta a}{a} - \frac{2\Delta P}{P}\right),$$

hence:

$$\frac{\Delta P}{P} = \frac{3}{2}\frac{\Delta a}{a}.$$

Here, we used the fact that $(1+x)^n \approx 1 + nx$ for $x \ll 1$.

Therefore:

$$\begin{aligned}
\frac{\Delta P}{\Delta t} &= \frac{3P}{2a}\frac{\Delta a}{\Delta t} = -\frac{3}{2}\cdot\frac{64}{5}\frac{P}{a}\frac{G^3}{c^5}\frac{M_1M_2(M_1+M_2)}{a^3} = -\frac{96}{5}\frac{G^3}{c^5}\frac{P}{a}\frac{M_1M_2(M_1+M_2)}{G(M_1+M_2)P^2}\cdot 4\pi^2 \\
&= -\frac{96}{5}(2\pi)^2\frac{G^2}{c^5}\frac{M_1M_2}{aP} = -\frac{96}{5}(2\pi)^2\frac{G^2}{c^5}\frac{M_1M_2}{P}\frac{(2\pi)^{2/3}}{G^{1/3}(M_1+M_2)^{1/3}P^{2/3}} \\
&= -\frac{96}{5}(2\pi)^{8/3}\frac{G^{5/3}}{c^5}\frac{M_1M_2}{(M_1+M_2)^{1/3}}\frac{1}{P^{5/3}} = -\frac{96}{5}(2\pi)^{8/3}\frac{G^{5/3}}{c^5}\frac{M_1M_2}{(M_1+M_2)^{1/3}}\frac{1}{P^{5/3}} \\
&= -\frac{96}{5c^5}(2\pi)^{8/3}\left(\frac{G\mathcal{M}}{P}\right)^{5/3} = -\frac{192\pi}{5c^5}(G\mathcal{M})^{5/3}\left(\frac{P}{2\pi}\right)^{-5/3},
\end{aligned}$$

Thus, by knowing the orbital period and its rate of change from observations, we can determine the chirp mass:

$$\mathcal{M} = \left(\frac{5}{192\pi}\right)^{3/5}\frac{c^3}{G}\frac{P}{2\pi}\left(-\frac{\Delta P}{\Delta t}\right)^{3/5} = 0.319\,M_\odot,$$

and find the characteristic strain amplitude h.

Alternatively, if we notice that the characteristic strain amplitude h depends on $(G\mathcal{M})^{5/3}$ which we can get directly from the rate of change of the period:

$$(G\mathcal{M})^{5/3} = -\frac{\Delta P}{\Delta t}\left(\frac{P}{2\pi}\right)^{5/3}\frac{5c^5}{192\pi},$$

we can skip calculating the chirp mass and get the characteristic strain amplitude directly, which is all we need to check if the binary can be detected.

Either way, the gravitational wave strain is:

$$h = -\frac{5c}{192\pi^2}P\frac{\Delta P}{\Delta t}\frac{1}{D}.$$

If we plug in the numerical values, we get $h = 1.1 \times 10^{-22}$ and $S = 8.4 \times 10^{-20}$. Checking the plot, this is above the expected sensitivity of LISA at 5 mHz. Thus, this object should be detected by LISA.

Calculating the chirp mass or $(G\mathcal{M})^{5/3}$ as a function of P and $\Delta P/\Delta t$	15 points
Calculating h and S	5 points
Correct conclusion – the system may be detected with LISA	3 points

Part (b)

To determine the masses of both components M_1 and M_2 we need two simultaneous $M_1 - M_2$ relations. The expression for the chirp mass:

$$\mathcal{M} = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}},$$

gives us one relation, and we can calculate the chirp mass of the observed system from the rate of change of the period, if that was not already done in part (a):

$$\mathcal{M} = \left(\frac{5}{192\pi}\right)^{3/5}\frac{c^3}{G}\frac{P}{2\pi}\left(-\frac{\Delta P}{\Delta t}\right)^{3/5} = 0.319\, M_\odot.$$

The second relation can be obtained from the mass–radius relation for white dwarfs given in the table and Kepler's third law.

We are told that for one of the components (call it '1'),

$$a = \frac{R_1}{0.139}.$$

From Kepler's third law we know that:

$$M_1 + M_2 = \left(\frac{a}{1\,\mathrm{au}}\right)^3\left(\frac{P}{1\,\mathrm{yr}}\right)^2,$$

which will give us the mass of the second component M_2 from the mass of the first, M_1.

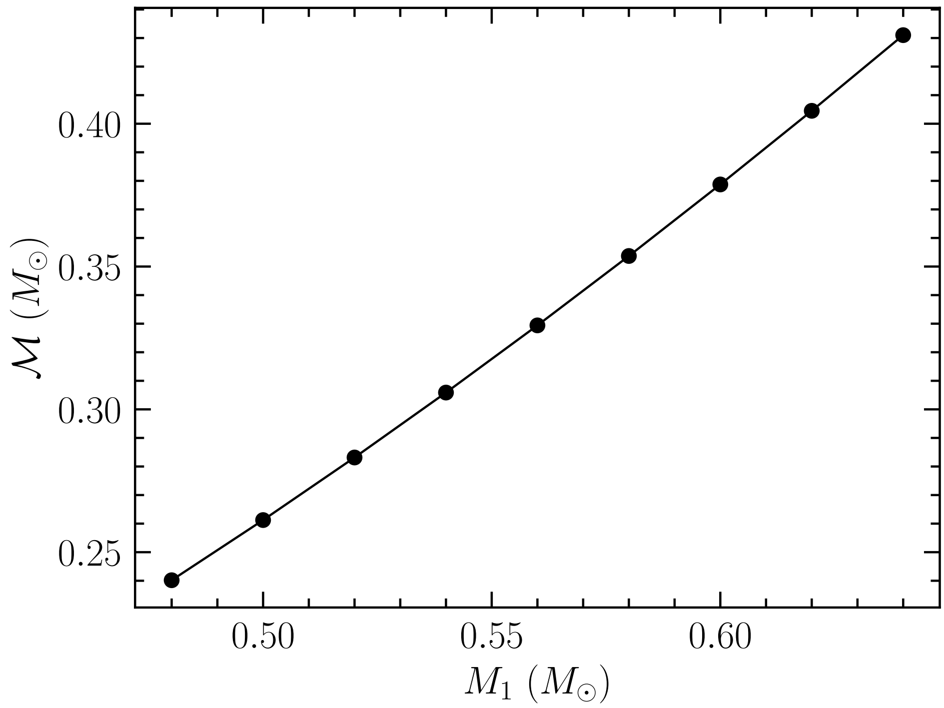
From here, it is not possible to derive an analytical formula for the masses of the components. Instead, we need to use numerical methods to estimate the result.

Taking the masses and radii listed in the given table as M_1 and R_1, we can obtain a and thus $M_1 + M_2$, M_2 and finally $\mathcal{M}$ for each mass:

M_1 ($M_\odot$)	R ($R_\odot$)	a ($R_\odot$)	$M_1 + M_2$	M_2	$\mathcal{M}$
0.48	0.0144	0.104	0.647	0.167	0.240
0.50	0.0147	0.106	0.689	0.189	0.261
0.52	0.0150	0.108	0.732	0.212	0.283
0.54	0.0153	0.110	0.776	0.236	0.306
0.56	0.0156	0.112	0.823	0.263	0.329
0.58	0.0159	0.114	0.871	0.291	0.354
0.60	0.0162	0.117	0.922	0.322	0.379
0.62	0.0165	0.119	0.974	0.354	0.405
0.64	0.0168	0.121	1.028	0.388	0.431

The actual chirp mass is $\mathcal{M} = 0.319\,M_\odot$. Therefore, by linear interpolation or graphically, we estimate $M_1 = 0.55\,M_\odot$ and $M_2 = 0.25\,M_\odot$.
(The student does not need to calculate $\mathcal{M}$ for all values of M_1.)

Graphical solution:



Calculating the chirp mass	5 points
Deriving two $M_1 - M_2$ relations	5 points
Determining the masses of both components	10 points