1 Solution: Gravitational wave astronomy (45 points)

1.1 Data from the graph

(1.1.1)
To make the scaling as accurate as possible, we take the time values far apart.
When $x = 0.0\,\text{cm}$ $t = -30\,\text{s}$, using the ruler we can also measure that at $t = 0\,\text{s}$ we have $x = 10.6\,\text{cm}$.
For range $\Delta x \geq 10cm$ 1 pt. For $6cm < \Delta x < 10cm$ 0.5 pt. For $3cm < \Delta x < 6cm$ 0.2 pt. For $\Delta x < 3cm$ 0pt.
This leads to

$$0.0\,\text{cm} = (-30\,\text{s})A + B$$
$$10.6\,\text{cm} = (0\,\text{s})A + B = B$$
$$A = \frac{10.6\,\text{cm}}{30\,\text{s}} = 0.35\,\text{cm/s} \qquad \textbf{0.5}$$
$$\therefore t = \frac{x - 10.6}{0.35}\text{s} \qquad \textbf{0.5}$$

(1.1.2)
Analogues to the previous scaling:
For range $\Delta y \geq 4.5cm$ 1 pt. For $3cm < \Delta y < 4.5cm$ 0.5 pt. For $1.5cm < \Delta y < 3cm$ 0.2 pt. For $y < 1.5cm$ 0pt..

$$0.0\,\text{cm} = C\log(30\,\text{Hz}) + D$$
$$5.3\,\text{cm} = C\log(500\,\text{Hz}) + D \qquad \textbf{1.0}$$
$$5.3\,\text{cm} = C\log\left(\frac{500}{30}\right)$$

$$C = 4.34\,\text{cm} \qquad \textbf{0.25}$$
$$D = -6.41\,\text{cm} \qquad \textbf{0.25}$$
$$\therefore f = 10^{\left(\frac{y+6.41}{4.34}\right)}\text{Hz} \qquad \textbf{0.5}$$

(1.1.3)
It is not necessary to use X-scaling as students can choose convenient points from the graph.
If has at least one value more that 100Hz 1.4pt.

x(cm)	y(cm)	t(s)	f(Hz)	$f^{-8/3}(Hz^{-8/3})$
-	0.5	-28.0	39.1	5.67E-05
-	0.5	-26.0	39.1	5.67E-05
-	0.6	-24.0	41.3	4.92E-05
-	0.6	-22.0	41.3	4.92E-05
-	0.7	-20.0	43.5	4.27E-05
-	0.8	-18.0	45.9	3.71E-05
-	0.9	-16.0	48.4	3.22E-05
-	1.0	-14.0	51.0	2.79E-05
-	1.1	-12.0	53.8	2.43E-05
-	1.2	-10.0	56.7	2.11E-05
-	1.4	-8.0	63.1	1.59E-05
-	1.6	-6.0	70.1	1.20E-05
-	1.8	-4.0	78.0	0.90E-05
-	2.3	-2.0	101.7	0.43E-05
-	3.4	0.0	182.4	0.09E-05

0.4 points for each row (except last column). Maximum 7 points.

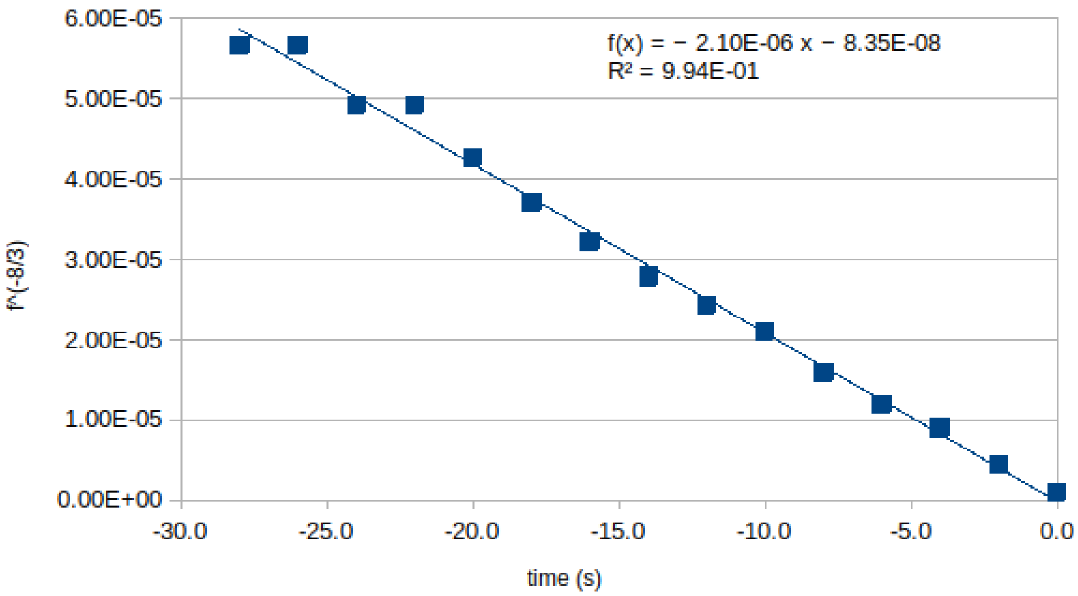
1.2 Calculate system parameters

(1.2.1)

$$\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3} = f^{-11/3}\dot{f} \qquad \mathbf{1.0}$$

$$\therefore \left[\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3}\right]t + \left[\frac{3}{8}\right]f^{-8/3} + C = 0 \qquad \mathbf{2.0}$$

(1.2.2)



Calculation of $f^{-8/3}$ for each value 3/14 values **3.0**

Graph:

Indicated X axis, written few numbers on the axis and written proper units 1pt. Indicated Y axis, written few numbers on the axis and written proper units 1pt. Used $70-100$ percent of the graphing paper 2pt, for $50-70$ percent 1.5pt, for $30-50$ percent 1pt, for $30-50$ percent 0.5pt. For each correct point (within 2mm distance from the correct positions) on the graph 0.35pt

$$\text{slope} = -2.10\times10^{-6} \qquad \mathbf{1.1}$$

$$\text{slope} = \frac{8}{3}\left[\frac{96}{5}\pi^{8/3}\left(\frac{GM_{chirp}}{c^3}\right)^{5/3}\right]$$

$$M_{chirp} = \left[\left(\frac{5}{256}\right)^{3/5}\frac{c^3}{G}\pi^{-8/5}\right](\text{slope})^{3/5} \qquad \mathbf{2.0}$$

$$\therefore M_{chirp} = 2.39\times10^{30}\,\text{kg} = 1.20M_{\odot} \qquad \mathbf{2.0}$$

$$\Delta\text{slope} = 9\times10^{-8} \qquad \mathbf{2.0}$$

$$\frac{\Delta M_{chirp}}{M_{chirp}} = \frac{3}{5}\frac{\Delta\text{slope}}{\text{slope}} = 0.023 \qquad \mathbf{2.0}$$

$$\Delta M_{chirp} = 0.03M_{\odot} \qquad \mathbf{1.0}$$

For uncertainty 70 percent points for the method and 30 percent for the result.

(1.2.3)

$$M_{chirp} = \frac{M_{chirp_{\text{detactor}}}}{1+z} = \frac{1.20M_{\odot}}{1.009783} = 1.19M_{\odot} \qquad \mathbf{2.0}$$

(1.2.4)

$$M_{chirp} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

$$\therefore M_{chirp} = m_2 \sqrt[5]{\frac{q^3}{1+q}}$$ **2.0**

This is an increasing function of q

$$\therefore m_{2,min} = M_{chirp} 2^{1/5} = 1.37 M_\odot$$

$$m_{2,max} = 1.60 M_\odot$$ **1.0**

$$\text{Also, } M_{chirp} = \frac{m_1}{q} \sqrt[5]{\frac{q^3}{1+q}} = \frac{m_1}{\sqrt[5]{q^2(1+q)}}$$ **2.0**

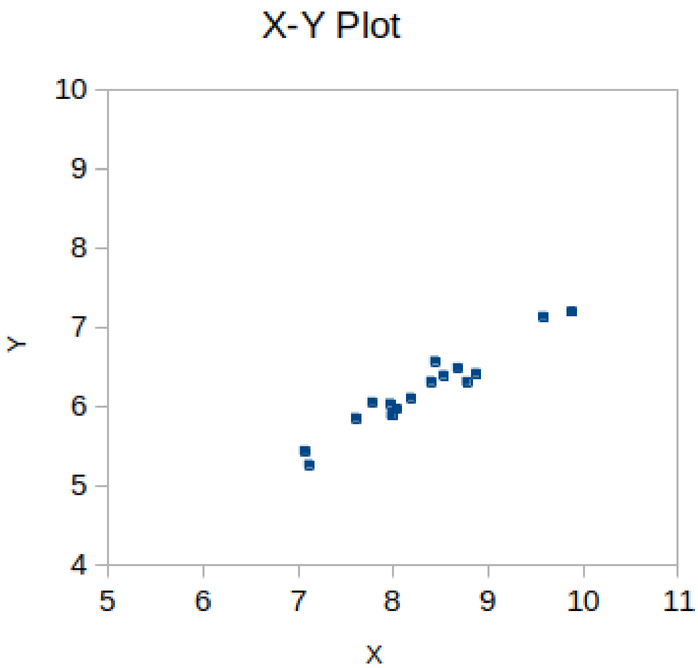
$$\therefore m_{1,max} = M_{chirp} 2^{1/5} = 1.37 M_\odot$$

$$m_{2,min} = 1.17 M_\odot$$ **1.0**

2 Solution: Galactic Surveys (105 points)

Part 1

Part (a)



Indicated X axis, written few numbers on the axis and written proper units 0.5pt. Indicated Y axis, written few numbers on the axis and written proper units 0.5pt. Used $70-100$ *percent of the graphing paper 2pt, for* $50-70$ *percent 1.5pt, for* $30-50$ *percent 1pt, for* $30-50$ *percent 0.5pt. If drawn* $14-16$ *points 3pt, If* $10-14$ *then 2pt, for 6-10 1pt and for* $0-6$ *0pt*

Part (b)

The normal to the disk also lies in XY plane. The line of sight (which we can take as X' axis) is also perpendicular to the normal. Thus, it can be calculated by normalizing $8.300\hat{\imath}+6.200\hat{\jmath}+1.100\hat{k}$. So

$$\hat{x}' = 0.79668\hat{\imath} + 0.59511\hat{\jmath} + 0.10558\hat{k} \qquad \mathbf{1.0}$$

(here the coefficient of $\hat{k}$ is unimportant). Z' axis lays in XY plane and is perpendicular to X'. So

$$\begin{aligned}\vec{z'} &= \hat{x}' \times \hat{k} \\ &= 0.59511\hat{\imath} - 0.79668\hat{\jmath} && \mathbf{2.0} \\ |\vec{z'}| &= 0.99441 && \mathbf{0.5} \\ \therefore\ \hat{z}' &= 0.59845\hat{\imath} - 0.80116\hat{\jmath} && \mathbf{0.5}\end{aligned}$$

[Alternative way: Calculate the equation of line of galaxies using linear regression 2pt, construct perpendicular line 1pt, normalise 1pt]

Part (c)

$$\begin{aligned}\hat{x}' &= 0.79668\hat{\imath} + 0.59511\hat{\jmath} + 0.10558\hat{k} \\ \hat{z}' &= 0.59845\hat{\imath} - 0.80116\hat{\jmath} \\ \hat{y}' &= \hat{z}' \times \hat{x}' \\ &= -0.084589\hat{\imath} - 0.063187\hat{\jmath} + 0.99441\hat{k} && \mathbf{2.0}\end{aligned}$$

Part (d)

Each coordinate in $X'Y'Z'$ system can be found by following equations

$$x' = \hat{x}' \cdot (\vec{r} - \vec{r_c})$$
$$y' = \hat{y}' \cdot (\vec{r} - \vec{r_c})$$
$$z' = \hat{z}' \cdot (\vec{r} - \vec{r_c})$$

8/3 points for each formula, here, $\vec{r}$ is coordinate of a satellite galaxy and $\vec{r_c}$ is coordinate of the central galaxy.
These values are calculated in the following table (**1.0** for each row):

	$x - x_c$	$y - y_c$	$z - z_c$	x'	y'	z'
1	0.101	0.109	0.294	0.1766	0.2769	-0.02688
2	1.583	0.989	0.406	1.893	0.2073	0.1550
3	0.583	0.213	1.126	0.7101	1.057	0.1783
4	0.144	0.369	1.339	0.4757	1.296	-0.2094
5	-0.518	-0.152	-1.458	-0.6571	-1.396	-0.1882
6	0.480	0.105	1.363	0.5888	1.308	0.2031
7	-0.310	-0.319	-0.568	-0.4968	0.5184	0.07005
8	0.240	0.188	-0.731	0.2259	0.7591	-0.006988
9	-0.325	-0.170	0.116	-0.3478	0.1536	-0.05830
10	-1.228	-0.769	-0.612	-1.500	-0.4561	-0.1188
11	-0.263	-0.235	-0.485	-0.4006	-0.4452	0.03088
12	0.381	0.283	-0.366	0.4333	-0.4141	0.001284
13	-1.185	-0.941	0.303	-1.4721	0.4610	0.04472
14	1.287	0.930	-0.778	1.4966	-0.9413	0.02514
15	-0.107	-0.096	-0.185	-0.162	-0.169	0.0129
16	-0.687	-0.353	0.566	-0.6976	0.6433	-0.1283

Part (e)

Spherical coordinates can by calculated by following relations:

$$R = \sqrt{x'^2 + y'^2 + z'2} \quad \mathbf{1.0}$$
$$\phi = \tan^{-1}\left(\frac{y'}{x'}\right) \quad \mathbf{1.0}$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right) \quad \mathbf{1.0}$$

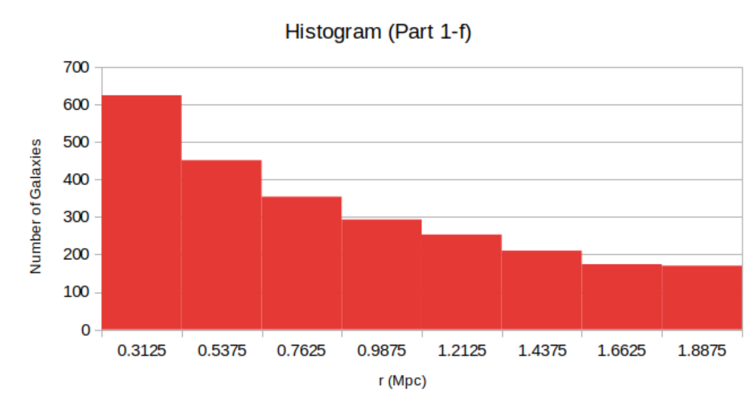
Calculated values are given in the following table (**0.5** for each row):

	R	ϕ	ϕ (deg)	θ
1	0.3294	1.004	57.51°	1.652
2	1.910	0.1091	6.25°	1.490
3	1.286	0.9792	56.10°	1.432
4	1.396	1.219	69.84°	1.721
5	1.555	4.273	244.80°	1.692
6	1.449	1.148	65.77°	1.430
7	0.7214	3.948	226.22°	1.474
8	0.7920	5.002	286.57°	1.580
9	0.3847	2.726	156.18°	1.723
10	1.573	3.437	196.91°	1.646
11	0.5997	3.980	228.02°	1.519
12	0.5993	5.520	316.30°	1.569
13	1.543	2.838	162.61°	1.542
14	1.768	5.722	327.83°	1.557
15	0.2343	3.948	226.20°	1.516
16	0.9576	2.397	137.32°	1.705

Part (f)

To divide the range in 8 equal bins, we must take $r_{bin} = 0.225\,\text{Mpc}$. **1.0**
By calculating bin ranges and counting the number of satellite galaxies in each bin, we get: (**0.5** for each row)

	Bin Center	Bin range	n
1	0.3125 Mpc	0.200 Mpc–0.425 Mpc	624
2	0.5375 Mpc	0.425 Mpc–0.650 Mpc	451
3	0.7625 Mpc	0.650 Mpc–0.875 Mpc	354
4	0.9875 Mpc	0.875 Mpc–1.10 Mpc	293
5	1.2125 Mpc	1.10 Mpc–1.325 Mpc	253
6	1.4375 Mpc	1.325 Mpc–1.55 Mpc	210
7	1.6625 Mpc	1.55 Mpc–1.775 Mpc	174
8	1.8875 Mpc	1.775 Mpc–2.00 Mpc	170



For plotted histogram **0.5** for each data point.

Part (g)

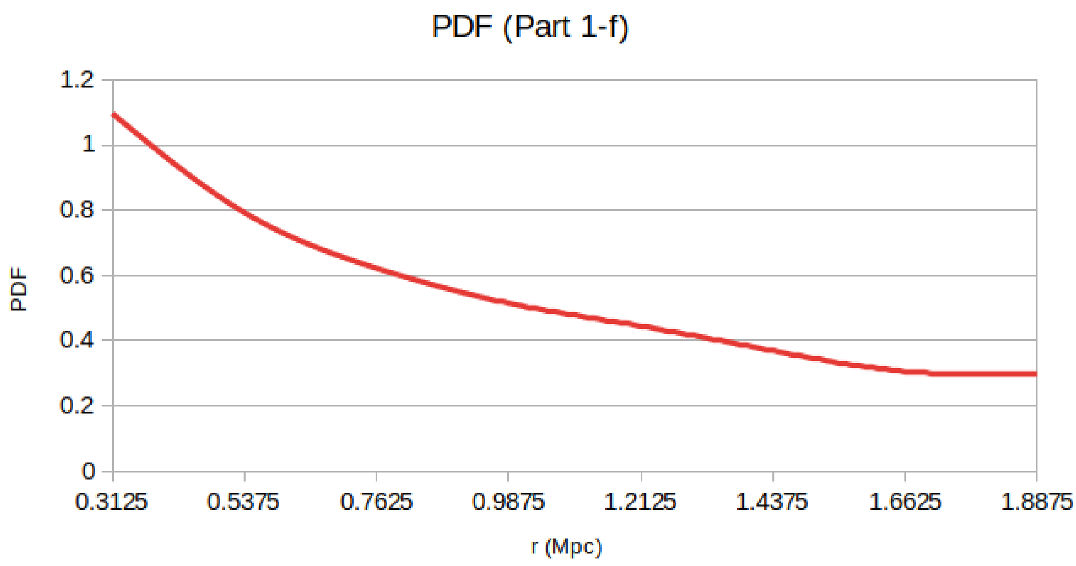
The sum of the area under the histogram is,

$$\begin{aligned} I &= \sum_{i=1}^{8} n_i \cdot r_i \\ &= r_{bin} \sum_{i=1}^{8} n_i \\ &= 0.225 \times 2529 \\ I &= 569.025 \end{aligned}$$ **2.0**

After dividing each n_i by I we get (**0.25** for each row):

	Bin Center	n_i	Probability Density
1	0.3125 Mpc	624	1.097
2	0.5375 Mpc	451	0.793
3	0.7625 Mpc	354	0.622
4	0.9875 Mpc	293	0.515
5	1.2125 Mpc	253	0.445
6	1.4375 Mpc	210	0.369
7	1.6625 Mpc	174	0.306
8	1.8875 Mpc	170	0.299

Plotting this data gives us (no separate points)



Part 2

Part (a)

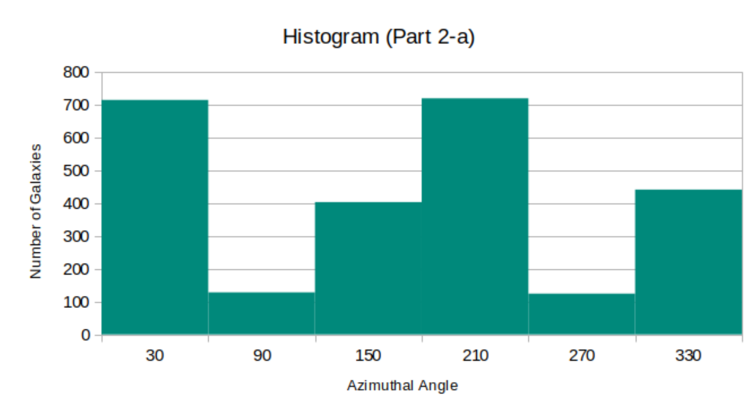
Making 60° bins for ϕ (**0.5** for each row),

	Center of the bin	Number of galaxies
1	30	714
2	90	128
3	150	403
4	210	719
5	270	124
6	330	441

The expected value for each bin, in case of uniform distribution is $\bar{n} = 2529/6 = 421.5$. Thus, the standard deviation will be

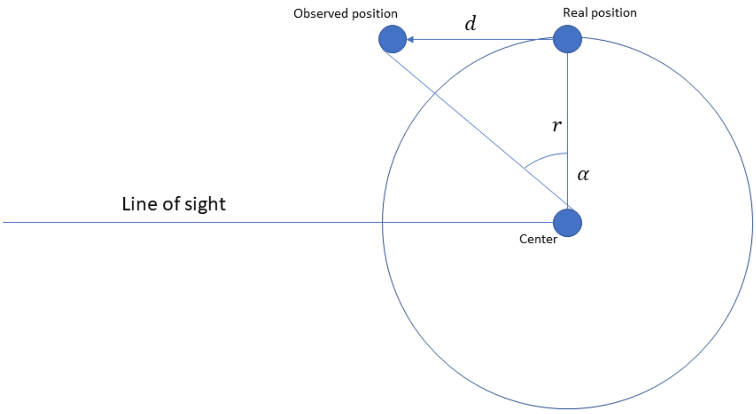
$$\sigma_1 \approx 264$$ **2.0**

The histogram looks like (no separate points)



Part (b)

A galaxy with the orbital radius r around central mass M is shown in the drawing



$$v_{\text{orbital}} = \sqrt{\frac{GM}{r}} \qquad \textbf{1.0}$$

$$d_{\text{uncorrected}} = \frac{v_{\text{measured}}}{H_0} = \frac{(v_{\text{redshit}} + v_{\text{orbital}})}{H_0}$$

$$\therefore \Delta d = \frac{v_{\text{orbital}}}{H_0} = \frac{1}{H_0}\sqrt{\frac{GM}{r}} \qquad \textbf{1.5}$$

$$\Delta\phi = \tan^{-1}\left(\frac{\Delta d}{r}\right) = \tan^{-1}\left(\frac{1}{H_0}\sqrt{\frac{GM}{r^3}}\right) \qquad \textbf{1.5}$$

[Alternative Solution:

$$v_{orb} = \sqrt{\frac{GM}{r}} \qquad \textbf{1.0}$$

$$d = \frac{v}{H_0} = \frac{v_{rredshift} - v_{orb}cos(\phi)}{H_0} \qquad \textbf{1.0}$$

$$\Delta d = r \cdot sin(\Delta\phi) = \frac{v_{orb} \cdot cos(\Delta\phi)}{H_0} \qquad \textbf{1.0}$$

$$tan(\Delta\phi) = \frac{v_{orb}}{r \cdot H_0} \qquad \textbf{0.5}$$

$$\Delta\phi = tan^{-1}\left(\frac{v_{orb}}{r \cdot H_0}\right) \qquad \textbf{0.5}$$

]

Part (c)

We calculate $\Delta\phi$ for each value of r and add it to the respective ϕ (choice of addition / subtraction is dependendent on sign convention. Students should realise that some of the galaxies should get added to bins 2 and 5).
The table looks like this (**0.25** for each row)

	ϕ_1	$\Delta\phi$	$\phi_{\text{corrected}}$	Number of galaxies
1	57.51°	27.36°	83.87°	291
2	6.25°	2.03°	8.28°	170
3	56.10°	3.68°	59.78°	253
4	69.84°	3.25°	73.09°	8
5	244.80°	2.77°	247.57°	72
6	65.77°	3.08°	68.84°	120
7	226.22°	8.69°	234.92°	302
8	286.57°	7.57°	294.14°	52
9	156.18°	21.44°	177.62°	28
10	196.91°	2.72°	199.63°	40
11	228.02°	11.41°	239.43°	72
12	316.30°	11.42°	327.72°	379
13	162.61°	2.80°	165.41°	82
14	327.83°	2.28°	330.11°	62
15	226.20°	39.56°	265.77°	305
16	137.32°	5.71°	143.03°	293

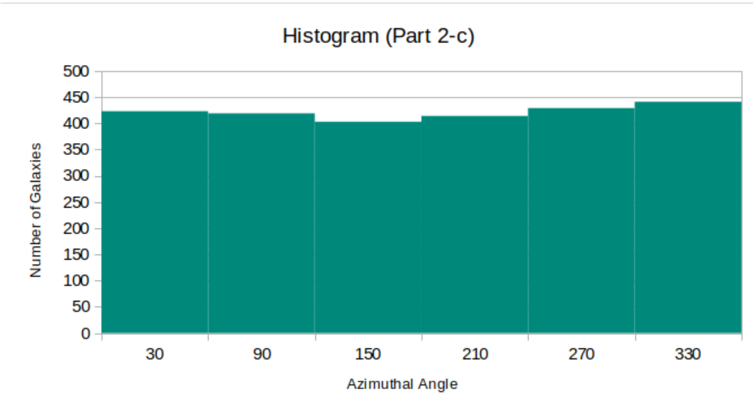
New corrected distribution looks like this (**0.5** for each correct n):

	Center of the bin	Number of galaxies
1	30	423
2	90	419
3	150	403
4	210	414
5	270	429
6	330	441

Thus, the standard deviation will be

$$\sigma_1 \approx 13$$ **1.0**

The histogram looks like (no separate points)



Part 3

Part (a)

With simple combinatorics one can get:

$$N_{CC} = \frac{1}{2}N_C(N_C-1) = 124750$$ **1.0**

$$N_{CS}^* = N_C N_S = 500000$$ **1.0**

$$N_{CS} = N_C(N_C-1)N_S = 249500000$$ **1.0**

$$N_{SS}^* = \frac{1}{2}N_C N_S(N_S-1) = 249750000$$ **1.0**

$$N_{SS} = \frac{1}{2}N_C(N_C-1)N_S^2 = 124750000000$$ **1.0**

Part (b)

We can calculate all probability densities using center-center and center-satellite densities. Non-normalized values (with $c_1 = c_2 = c_3 = 1$) (**1 pt**) are calculated in the following table (**0.5** for each correct value):

	r	ρ^*_{CS}	ρ_{CC}	ρ^*_{SS}	ρ_{CS}	ρ_{SS}
1	0.3125	1.097	0.406	0.376	0.446	7.931
2	0.5375	0.793	0.808	0.338	0.454	5.934
3	0.7625	0.622	1.341	0.295	0.452	3.951
4	0.9875	0.515	1.134	0.262	0.382	3.913
5	1.2125	0.445	0.479	0.240	0.292	4.899
6	1.4375	0.369	0.143	0.196	0.209	4.619
7	1.6625	0.306	0.073	0.155	0.161	3.774
8	1.8875	0.299	0.060	0.168	0.173	4.111

For normalization, we get $c_1 = 2.19; c_2 = 1.73; c_3 = 0.114$. (**2 pt** for each)
Multiplying relevant densities by this values gives (**0.5** for each correct value):

	r	ρ^*_{SS}	ρ_{CS}	ρ_{SS}
1	0.3125	0.823	0.771	0.901
2	0.5375	0.739	0.785	0.674
3	0.7625	0.646	0.781	0.449
4	0.9875	0.573	0.662	0.444
5	1.2125	0.525	0.504	0.556
6	1.4375	0.429	0.362	0.525
7	1.6625	0.340	0.279	0.429
8	1.8875	0.369	0.300	0.467

Part (c)

Total probability density is weighted sum of all densities:

$$\rho \sim \frac{N_{CC}\rho_{CC} + N^*_{CS}\rho^*_{CS} + N_{CS}\rho_{CS} + N^*_{SS}\rho^*_{SS} + N_{SS}\rho_{SS}}{N_{CC} + N^*_{CS} + N_{CS} + N^*_{SS} + N_{SS}}$$ **2.0**

Final PDF is given in the following table (**0.5** for each value):

	r	ρ
1	0.3125	0.900
2	0.5375	0.674
3	0.7625	0.450
4	0.9875	0.445
5	1.2125	0.556
6	1.4375	0.524
7	1.6625	0.428
8	1.8875	0.466

Note: The constant of normalisation in this case turns out to be 1.00. Hence no need to separately normalise this PDF. **1.0**
Most similar to this density is **plot E** on Figure 1 of question paper. **1.0**