V1

Short theoretical questions – solutions

Each question max 10 points

Question 1

Perihelium distance is much smaller than aphelium distance.

Thus, semimajor axis of the comet orbit $a \approx 17500$ AU. **[1]**

Application of the III Kepler's Law to get the comet orbital period **[4]**

Actual calculations: $T = a^{2/3} \approx 2.32 \cdot 10^6$ (T in years) **[4]**

The sought solution is equal to one half of T, i.e. approx. $1.16 \cdot 10^6$ years **[1]**

Question 2

$v_e = \sqrt{(2GM_{sol}\, n/R)}$, **[5]**

solve for n. (~ 100 000). **[5]**

Question 3

Sum of potential and kinetic energy of Voyager is $E = -G\, m\, M_\odot / r + mv^2 / 2$ **[3]**

Where m – mass of Voyager, $r = 116.406$ AU, $v = 17.062$ km/s.

Substituting numerical values into energy equation, **[3]**

we get $E > 0$, therefore the orbit is hyperbolic **[1]**

Brightness of the Sun is $m(r) = m_\odot + 5 \log(r)$ **[2]**

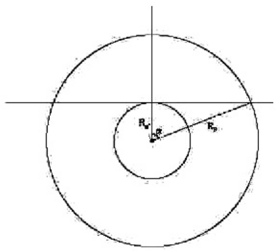
≈ -16.5 mag. **[1]**

Question 4
The orbit of Phobos and the rotation of the planet are in the same direction, so the angular velocity of the moon relative to Mars, $\omega_{P/M}$, is equal to the difference
$\omega_{P/M} = \omega_M - \omega_P$ **[2]**

where ω_M and ω_P are the angular rotation velocity of Mars and the angular orbital velocity of Phobos respectively.
For ω_M we have $\omega_M = 2\pi / T_M$ where T_M is the period **[1]**

The time above the horizon, t, is found from the relation (see drawing):
$t = 2\alpha / \omega_{P/M}$ **[2]**

where $\alpha = R_M / R_P$. **[1]**



Find the angular velocity of Phobos ω_P assuming gravity is equal to the centripetal force:

$$\omega_P = \mathrm{sqrt}(\, GM_M / R^{3/2}_P) \qquad \textbf{[2]}$$

Substituting:
$\omega_M = 7.0882 \cdot 10^{-5}\ s^{-1}$,
$\omega_P = 2.2784 \cdot 10^{-4}\ s^{-1}$,
$\omega_{P/M} = -1.5696 \cdot 10^{-4}\ s^{-1}$
where "–" means that the moon "leads" the rotation of the planet; substitute the absolute value into the equation,

thus
$\alpha = 68^{\circ}.794 = 1.2007$ rad.
and $t \approx 15300\ s = 4h\ 15m$. **[2]**

Question 5

Angular resolution: $\sin(\lambda) = 1.22\ \lambda / D$ [2]

$D_{radio}[m] = (\lambda_{radio} \cdot D_{optic}) / \lambda_{optic} = (1cm \cdot 10cm)/550nm = (0.01\ m \cdot 0.1\ m) / 0.00000055\ m = 1820\ m$

equation of angular resolution as a function of λ and D [2]

proportion of $\lambda_{radio}/D_{radio} = \lambda_{optic}/D_{optic}$ [2]

proper calculation [2]

and result assuming λ_{optic} in proper range: 500 – 600 nm [2]

400 – 900 nm

out of range

Question 6

Calculation of the moment of inertia of Earth [2]

The moment of inertia is $I = \frac{2}{5} m_E R_E^2$

Equation of motion for this case $K = I \cdot \varepsilon$ [2]

where ε is angular acceleration and K is moment of force.

Calculation of angular acceleration [1]

Acceleration is equal to $\varepsilon = K/I$. From the assumption, the acceleration ε is constant and we find its numerical value equal to $6 \cdot 10^{22}\ s^{-2}$.

Calculation of the change of ε during assumed period of time [1]

Multiplying this value by $1.9 \cdot 10^{16}$ s [600 My] we obtain $\Delta\omega = 1.14 \cdot 10^{-5}\ s^{-1}$.

Calculation of the angular frequency in the past [2]

Using the definition of the angular frequency one can calculate its recent value $2\pi/86164\ s^{-1} = 7.29 \cdot 10^{-5}\ s^{-1}$. Subtracting from it calculated above $\Delta\omega$ one gets $\omega_{past} = 6.15 \cdot 10^{-5}\ s^{-1}$

Calculation of the day length in the past and number of days in the ancient year [2]

Having angular frequency one can get ancient period of Earth rotation . Dividing sidereal year length by the obtained period one gets number of days in the ancient year equal to about 420 days.

Question 7

Limiting ourself to the orbits with peri and apogeum in the explosion point, the angle β should be equal to 90° or 270°, while the angle α should be:

1). $\alpha \in <0° ; 90°)$ or $\alpha \in (270° ; 360°>$, [2]

2). $\alpha = 90°$ or $\alpha = 270°$, [2]

3). $\alpha \in (90° ; 120°)$ or $\alpha \in (240° ; 270°)$, [2]

4). $\alpha = 120°$ or $\alpha = 240°$, [2]

5). $\alpha \in (120° ; 180°)$ or $\alpha \in (180° ; 240°)$, [2]

respectively.

Taking into account other kind of orbits (without peri and apogeum) one has to make the angle β free (unfixed).

Question 8

Gravitational force can be expressed in the form: $F_g = \frac{4}{3}\pi \cdot r^3 \cdot \rho \cdot G \cdot \frac{M}{R^2}$ **[2]**

The force of radiation reads $F_r = \pi \cdot r^2 \cdot C_0/c$ **[2]**
where r, ρ, R, G, M, C_0 are a radius and density of dust, distance form the Sun, gravitational constant, Solar mass and Solar constant respectively.
Comparing these forces one obtains $F_g = F_r$ **[2]**

and finally one gets the dust grain radius $r = \frac{3}{4}\frac{C_0 \cdot R^2}{c \cdot \rho \cdot G \cdot M}$ **[2]**

Substituting numerical values one obtains $r \approx 5 \cdot 10^{-7}$ m **[2]**

Question 9

Finding that surface brightness is independent of distance **[2]**

Relationship between m and the Sun surface brightness:
$m - \mu_{\odot} = -2.5 \log\ ((\pi \cdot \theta^2/4)\ /\ 1\ \text{arcsec}^2)$. **[2]**

Calculations of $\mu_{\odot} = -10.79$ **[1]**

Galaxy vs. Sun surface brightness (in magnitudes·arcsec^{-2})
$\mu - \mu_{\odot} = 18.0 + 10.79 = 28.79$ **[1]**

Relationship between the sought sky fraction covered by stars, x, and the surface brightness ratio: $\mu - \mu_{\odot} = -2.5 \log x$ **[3]**

Calculations $x = 10^{-0.4(\mu - \mu_{\odot})} \approx 3.1 \cdot 10^{-12}$ **[1]**

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Question 10

Formulation of a penetration conditions **[2]**

From the conditions specified in the exercise text the particle should be pass in the belt of magnetic field around quarter of the circle. If this particle leaves the belt the particle path should be a little longer. It means that the radius of this circle should be larger then the belt thickness.

Determination of the forces (centrifugal and magnetic) acting upon the particle, writing of the appropriate formulae **[4]**

Comparison of the forces and calculations **[3]**

Comparing centrifugal and magnetic force one obtains

$e \cdot v \cdot B = M \cdot v^2 / r$

So $e \cdot r \cdot B = p$, where p denotes the momentum. From the assumed approximation $e \cdot r \cdot B = E / c$

Thus definitively particle energy is described by $E = c \cdot e \cdot r \cdot B$.

Substituting numerical values one obtains $E \approx 3 \cdot 10^{-8}$ J it means $2 \cdot 10^2$ GeV

Answer and conclusion: **[1]**

Question 11

Within assumed cosmological model the following relation is valid:

$(1+z) = \dfrac{R_0}{R(z)} = \left(\dfrac{t_0}{t(z)}\right)^{2/3}$, where R_0 and $R(z)$ denote the current value of the scale factor and that for the redshift equal to z respectively, while t_0 and t(z) stands for the time that passed from the Big Bang. **[5]**

For z=6.03 the appropriate time $t(z) = 7.35\ 10^8$ years. Subtracting from $t(z)$ the interval equal to 560 My and 600 My one obtains $t_1 = 1.75 \cdot 10^8$ y and $t_2 = 1.35 \cdot 10^8$ y **[3]**

what is equivalent to $z_1 = 17.3$ and $z_2 = 20.8$. **[2]**

Question 12

At Northern Hemisphere declination of the non-rising stars $\delta < \varphi - 90^\circ$, where φ stands for latitude. So in Krakow $\delta < -39.9^\circ$. **[2]**

Within assumed precession model declination of the star can differ from ecliptic latitude β not more than $\varepsilon = +23.5^\circ$. So the ecliptic latitude β of the stars at the study should be determined. **[2]**

Solving spherical triangle

$\sin\beta = \cos\varepsilon \cdot \sin\delta - \sin\varepsilon \cdot \cos\delta \cdot \sin\alpha$ **[3]**

Substituting by numerical values for the stars at the study one obtains:

Sirius $\beta = -39^\circ.7$

Canopus $\beta = -75^\circ.9$ **[1]**

Due to the precession, declination of Sirius can reach -63.2° so it will become non rising star.

As far as Canopus is concern its declination never exceed -52.4°, so it will never be visible. **[2]**

Question 13

From the attached table it can be read that northern galactic pole has the following coordinates:

$$\alpha_G = 12^h 51^m;\ \delta_G = +27^\circ 08'.$$

It means that at the sidereal time $\theta = 0^h\ 51^m$ this pole is in lower culmination at the northern side and below the horizon. **[2]**

The angle between galactic equatorial plane and the horizon is equal to

$90^\circ - \delta_G + 90^\circ - \varphi = 103^\circ 18'$ and this angle is an analogue of ε in the equation of the ecliptic.

Analogue of R.A. in the equation of the ecliptic is $90^\circ - A$ because A is clockwise while R.A. is counterclockwise and at the intersection of the horizon plane and galactic equatorial azimuth is equal to 90°. **[2]**

So we get $tg(h) = \sin(90° - A)\, tg(180° - \varphi - \delta_G)$. So,
$h = \text{arc tg}\,(\cos A \cdot tg\,(-\varphi - \delta_G) = \text{arc tg}\,(-\cos A \cdot tg\,76°42')$. **[4]**

Checking:

$A = 0$, $h = -76°42'$,
$A = 90$, $h = 0°$. **[2]**

Question 14
Solar constant (from table) is $1366\ W\ m^{-2}$
Each reaction gives $26.8\ MeV = 26.8 \cdot 1.602 \cdot 10^{-13}\ J = 4.3 \cdot 10^{-12}\ J$ **[4]**
Thus there must be $1366 / 4.3 \cdot 10^{-12} = 3.2 \cdot 10^{14}$ reactions per second to produce this energy flux **[4]**
Each reaction produces 2 neutrinos, so there must be $6.4 \cdot 10^{14}$ neutrinos $m^{-2} \cdot s^{-1}$. **[2]**

Question 15
Correct choice of the wavelength: **[2]**

If the nature of the spectrum does not change, it is sufficient to perform the calculation for one wavelength. The simplest choice is for the wavelength of the maximum (Wien's law)

$\lambda = b/T$

Understanding definition of redshift **[2]**

From this
$z = (\lambda_a - \lambda_e)/\lambda_e$
where λ_a is the wavelength received at the Earth and λ_e is the emitted wavelength
thus $\lambda_e = \lambda_a/(z+1)$

Substituting into Wien's law, we find

$T = b \cdot (z+1) / \lambda_a$ **[4]**

Substituting the current value of the temperature we find the result **[2]**

Long theoretical questions – solutions and points

Question 1 – Transit of an exoplanet

Max 30 points

Calculation of the angular distance on the orbit which the planet travels during the transit **[6]**
Since the orbit is circular,

$$\sin\frac{\alpha}{2}=\frac{\pi t}{T}$$

where t and T are, respectively, the duration of the transit and the orbital period of the planet.

We can leave the angle as $\sin\frac{\alpha}{2}$ as this will be useful later; the numerical value of this expressiob is about 0.112.

Calculation of the velocity of the planet **[10]**
The velocity v at the beginning and end of the transit and the measured difference in velocities Δv form an isosceles triangle, with the angle between the equal sides = 2α .
Thus

$$2v\cdot\sin\frac{\alpha}{2}=\Delta v$$

from which we get a velocity equal to about 134 km/h. **[2]**

Calculation of the radius of the orbit **[2]**
since we know the velocity on a circular orbit, this is trivial:

$$2\pi r=vT\Rightarrow r=\frac{vT}{2\pi}$$

Substituting we get about $6.4\cdot10^6$ km. **[2]**

Calculation of the radius of the star **[2]**
With the velocity on the orbit and the transit duration, we get the radius of the star:

$2R=tv\Rightarrow R=\frac{tv}{2}$ = $7.5\cdot10^5$ km . The star is thus similar to the Sun in size. **[2]**

Calculation of the mass of the :

Given the orbital speed and radius, do this in the usual way:

$$\frac{v^2}{r}=\frac{GM}{r^2}$$ **[2]**

thus

$$M=\frac{v^2 r}{G}$$

The mass is therefore about $1,8\cdot10^{30}$ kg, so again similar to the mass of the Sun. **[2]**

Question 2 – Brightness of an elliptical galaxy

Max 30 points

1)

The luminosity distance is defined to satisfy a canonical inverse square law of the bolometric luminosity L (total emitted radiation power) and observed flux F (flux-luminosity relationship)

$$F = \frac{L}{4\pi d_L^2} \qquad (1)$$

Units of the flux and luminosity are expressed in SI ($W\cdot m^{-2}$ and W respectively), or CGS ($erg\cdot s^{-1}\cdot cm^{-2}$, $erg\cdot s^{-1}$ respectively).

[3]

2)

Only the part of the spectrum (namely B-band) is observed, but the inverse square law is applicable also to the monochromatic luminosity $L_\lambda(\lambda)$ and monochromatic flux $S(\lambda)$ [1], taking into account that observed wavelengths are not equal to emitted ones. (Observers frame is not the same as a frame of galaxy). For a small wavelength interval

$$S(\lambda_o)\cdot\Delta\lambda_o = \frac{L_\lambda(\lambda_e)}{4\pi d_L^2}\cdot\Delta\lambda_e,$$

Observed fraction of the total spectrum is redshifted into the frame of the observer, and a relationship between the observed and emitted wavelengths, λ_o and λ_e, can be expressed as

$$\lambda_e = \lambda_o/(1+z).$$

So,

$$S(\lambda)\Delta\lambda = \frac{1}{4\pi d_L^2}\cdot L_\lambda\left(\frac{\lambda}{1+z}\right)\cdot\frac{1}{1+z}\Delta\lambda. \qquad (2)$$

[9]

3)

The absolute luminosity is defined as the flux would be observed at a distance of 10 pc from the compact source at rest (not redshifted). It means that the frame would be the same as for the object.

$$S_{10}(\lambda) = \frac{L_\lambda(\lambda)}{4\pi d_{10}^2}, \qquad (3)$$

where d_{10} is the distance equal to 10 pc, $S_{10}(\lambda)$ denotes a monochromatic flux in the wavelength of λ measured at 10 pc from the object.

[3]

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[1] *Monochromatic flux is sometimes called as spectral density of flux (energy per unit time per unit area per unit wavelength) while monochromatic luminosity as spectral density of the luminosity (energy per unit time per unit wavelength). Their form is described by SED and thus spectral index.*

4)
Dividing eq. (2) by (3) we get

$$\frac{S(\lambda)}{S_{10}(\lambda)}=\left(\frac{d_{10}}{d_L}\right)^2\cdot\frac{L_\lambda\left(\frac{\lambda}{1+z}\right)}{L_\lambda(\lambda)}\cdot\frac{1}{1+z} \qquad (4)$$

for all the wavelengths.
The lower edge of validity of SED approximation is equal to 250nm. This corresponds to 1.5·250 nm=375 nm what is below the blue cut-off of the B filter(2), so SED approximation is applicable. Using the SED of the galaxy, we get

$$\frac{L_\lambda\left(\frac{\lambda}{1+z}\right)}{L_\lambda(\lambda)}\cdot\frac{1}{1+z}=\left(\frac{1}{1+z}\right)^4\frac{1}{1+z}=(1+z)^{-5} \qquad (5)$$

$$\frac{S(\lambda)}{S_{10}(\lambda)}=\left(\frac{d_{10}}{d_L}\right)^2\cdot(1+z)^{-5} \qquad (6)$$

The fraction of the monochromatic fluxes $S(\lambda)/S_{10}(\lambda)$ does not depend on the wavelength! So the relation (6) is valid also for blue band fluxes.

[9]

5)
The blue fluxes F and F_{10} have to be replaced by the corresponding magnitudes.
Using Pogson's Ratio it can be obtained:

$$m_B-M_B=-2.5\log\left(\frac{F}{F_{10}}\right)=5\log\left(\frac{d_L}{d_{10}}\right)+12.5\log(1+z)$$

$$M_B=m_B-5\log\left(\frac{d_L}{d_{10}}\right)+12.5\log(1+z) \qquad (7)$$

Substituting numerical data one can obtain:

$$M_B=20.40^{\mathrm{mag}}-5\log\left(2754\cdot10^5\right)-12.5\log(1.5)=-24.00^{\mathrm{mag}}$$

$$M_B=-24.00^{\mathrm{mag}} \qquad (8)$$

Please note, that precision is determined by m_B !

[3]

6)
Only the cD galaxies are so luminous, while the normal elliptical galaxies are substantially weaker. Accordingly, the galaxy is not a member of the cluster, but is a foreground object.

[3]

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[2] *Blue band : Effective Wavelength Midpoint λeff=445nm, FWHM=94nm [398nm÷492nm]*

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Question 3 – Proper motion

Max 30 points

Difference in radial distance is about $1{,}58 \cdot 10^{15}\, m$ **[2]**
Difference in right ascension is about 2.4767^{o} **[2]**
Difference in declination is about 1.84^{o} **[2]**

Thus the angular distance on the sky is about 3.09^{o} **[3]**

Calculating the linear distance perpendicular to the line of sight
Using trigonometric functions and the distance from the Sun, we obtain a distance of $2{,}1 \cdot 10^{15}\, m$ **[2]**

The distance between the stars is thus $2{,}63 \cdot 10^{15}\, m$ **[3]**

Calculate the difference in angulare velocity in R.A. and Dec. Calculations (same method) for R.A. and Dec give a difference of 0.18 arcsec/year in both. **[4]**

Conversion of angular velocity on the sky to transverse linear velocity. (works out aroud 15) **[4]**

Note that transverse velocity is a lower limit to total velocity **[2]**

Compare obtained velocity to velocity expected for the distances between stars.
At distances of tens of thousands of AU the velocity is too great for the stars to form a bound system even with very massive stars. Even for a pair of one light and one heavy star, the heavy one would have a mass of thousands of Solar masses. **[4]**

Final answer and conclusion: **[2]**