300 points for 3 problems, 100 points for each problem

I. Virgo Cluster

1. From table above, we have $\sum_i \frac{d_i}{s_i^2} = 56.1082$ and $\sum_i \frac{1}{s_i^2} = 3.5485$, so:

   $$d_{avg} = \frac{56.1082}{3.5485} \text{ Mpc} = 15.81 \text{ Mpc} \qquad \textbf{(30\%)}$$

2. The uncertainty (rms):

   $$s_d = \sqrt{\frac{1}{\sum_i \frac{1}{s_i^2}}} = \sqrt{\frac{1}{3.5485}} \text{ Mpc} = 0.53 \text{ Mpc}$$

   $$d_{Virgo} = 15.81 \pm 0.53 \text{ Mpc} \qquad \textbf{(10\%)}$$

3. Hubble constant

   $$H_0 = \frac{v}{d_{avg}} = \frac{1136 \text{ km/s}}{15.81 \text{ Mpc}} = 71.9 \text{ km/sec/Mpc} \qquad \textbf{(15\%)}$$

   Uncertainty:

   $$s_{H_0} = \frac{s_d}{d_{avg}} H_0 = \frac{0.53}{15.81} \times 71.85 = 0.0335 \times 71.85 = 2.4 \text{ km/sec/Mpc} \qquad \textbf{(15\%)}$$

   Hubble constant: $H_0 = 72 \pm 2 km/\text{sec}/Mpc$ **(5%)**

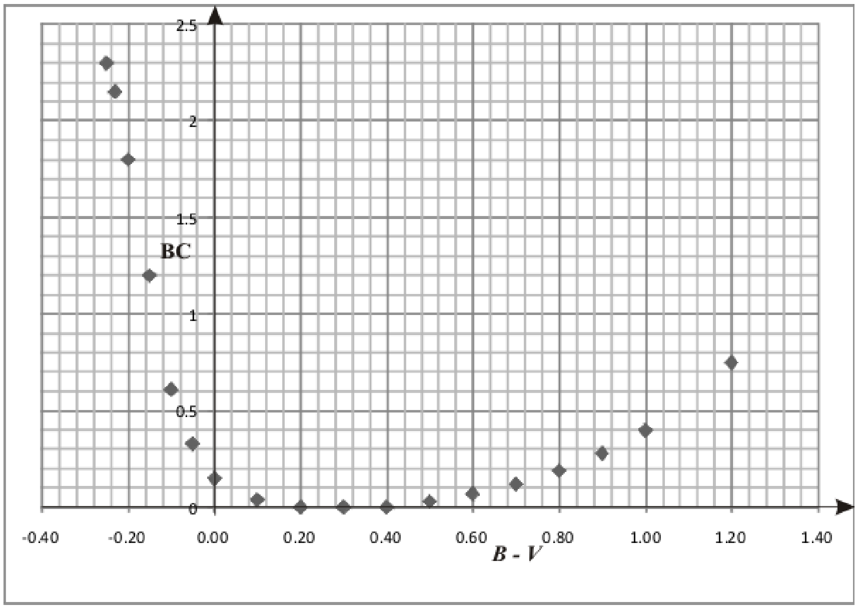
4. Hubble time:

   $$H_0^{-1} = \frac{1}{H_0} = \frac{1}{71.85 \text{ km/sec/Mpc}} \times 3.086 \times 10^{10} \text{ km/Mpc} = 4.29 \times 10^{17} \text{sec} = 13.6 \text{ Gyr} \qquad \textbf{(20\%)}$$

   Uncertainty is 3.35% which is 0.45 Gyr. **(5%)**

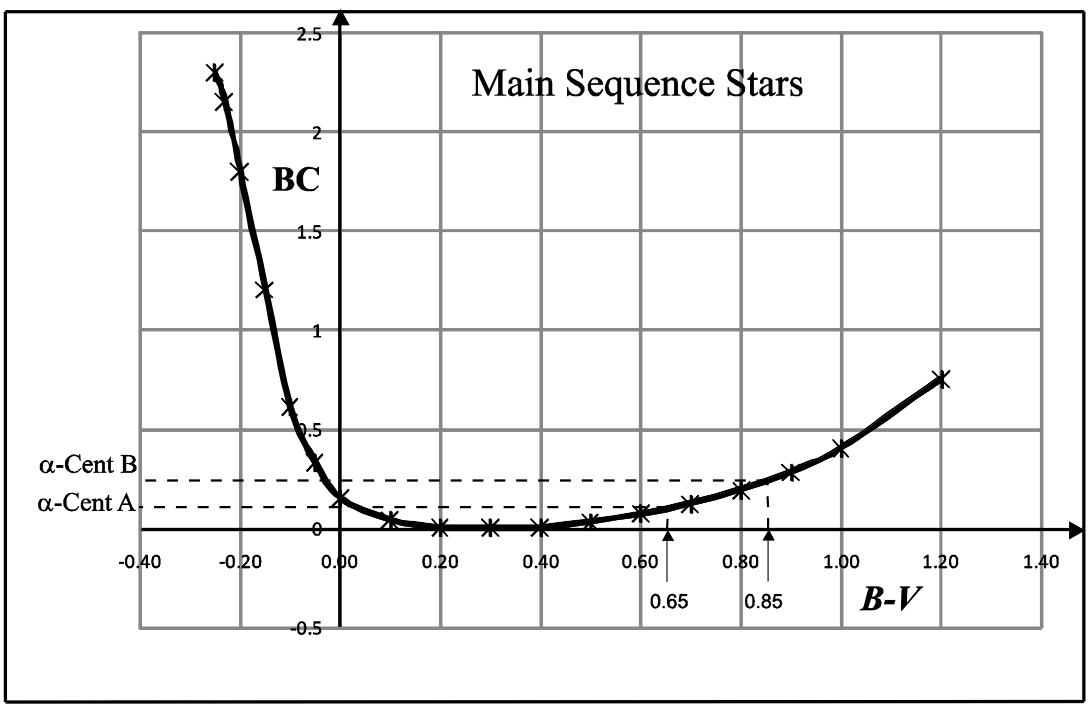
II. Determination of stellar masses in a visual binary system

1



(20%)

2.



⊙ For α Cen. A. , its colour index is given by B - V = 0.65. Then, by taking the line that intersecting the above curve we find BC = 0.10. **(10%)**

⊙ For α Cen. B. , its colour index is given by B - V = 0.85. Then, by taking the line that intersecting the above curve we find BC = 0.25. **(10%)**

From the relation $m_v - m_{bol} = BC$ it is then obtained,

- ⊙ For α Cen. A, m_v = -0.01 , we get m_{bol} = -0.01 – 0.10 = -0.11 **(10%)**
- ⊙ For α Cen. B, m_v = 1.34, we get m_{bol} = 1.34 – 0.25 = 1.09 **(10%)**

3. Periodic orbit of α Centauri star is P =79.24 years and its angle of semi major axis is α = 17.59". From the result 2a we find that the apparent bolometric magnitude of α Centauri A is m_{bolA} = -0.11, while the apparent bolometric magnitude of α Centauri B is m_{bolB} = 1.09. (two decimal places)

1st iteration :

Step 1 : Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be the mass of Centauri A and Centauri B stars, respectively. Then, we firstly take $\mathcal{M}_A + \mathcal{M}_B = 2$

Step 2 : Inserting the values of $\mathcal{M}_A + \mathcal{M}_B$, P, and α to the third Kepler equation namely,

$$\left(\frac{\alpha}{p}\right)^3 = (\mathcal{M}_A + \mathcal{M}_B)P^2$$

One gets p , that is

$$p = \left[\frac{\alpha^3}{(\mathcal{M}_A + \mathcal{M}_B)P^2}\right]^{1/3} = \left[\frac{(17.59)^3}{2(79.24)^2}\right]^{1/3} = [0.4334]^{1/3} = 0.7568''$$

Step 3 : We determine the magnitude of absolute bolometric using Pogson equation,

$$m_{bol} - M_{bol} = -5 - 5\log p \Longrightarrow M_{bol} = m_{bol} + 5 + 5\log p$$

For α Centauri A:

$$M_{bolA} = m_{bolA} + 5 + 5\log p = -0.11 + 5 + 5\log(0.7568)$$
$$= -0.11 + 5 - 0.6051 = 4.2849$$

For α Centauri B :

$$M_{bolB} = m_{bolB} + 5 + 5\log p = 1.09 + 5 + 5\log(0.7568)$$
$$= 1.09 + 5 - 0.6051 = 5.4849$$

Step 4 : We determine the mass of both stars using luminosity mass relation,

$$M_{bol} = -10.2\log\left(\frac{\mathcal{M}}{\mathcal{M}_\odot}\right) + 4.9 \Longrightarrow \mathcal{M} = 10^{\left(\frac{4.9 - M_{bol}}{10.2}\right)}\mathcal{M}_\odot$$

For α Centauri A:

$$\mathcal{M}_A = 10^{\left(\frac{4.9 - M_{bolA}}{10.2}\right)}\mathcal{M}_\odot = 10^{\left(\frac{4.9 - 4.2849}{10.2}\right)}\mathcal{M}_\odot = 1.1490\mathcal{M}_\odot$$

For α Centauri B:

$$\mathcal{M}_B = 10^{\left(\frac{4.9 - M_{bolB}}{10.2}\right)}\mathcal{M}_\odot = 10^{\left(\frac{4.9 - 5.4949}{10.2}\right)}\mathcal{M}_\odot = 0.8763\mathcal{M}_\odot$$

(30%)

2nd iteration Repeating the step 1 until 4 and using the previous resulting mass in the first iteration:

Step 1 : $\mathcal{M}_A + \mathcal{M}_B = 1.1490 + 0.8763 = 2.0253$

Step 2 : $$p = \left[\frac{\alpha^3}{(\mathcal{M}_A + \mathcal{M}_B)P^2}\right]^{1/3} = \left[\frac{(17.59)^3}{(2.0253)(79.24)^2}\right]^{1/3} = 0.7536$$

Step 3 : For α Centauri A:

$$M_{bolA} = m_{bolA} + 5 + 5\log p = -0.11 + 5 + 5\log(0.7536) = 4.2757$$

For α Centauri B :

$$M_{bolB} = m_{bolB} + 5 + 5\log p = 1.09 + 5 + 5\log(0.7536) = 5.4757$$

Step 4 : For α Centauri A:

$$\mathcal{M}_A = 10^{\left(\frac{4.9 - M_{bolA}}{10.2}\right)}\mathcal{M}_\odot = 10^{\left(\frac{4.9 - 4.2757}{10.2}\right)}\mathcal{M}_\odot = 1.1513\mathcal{M}_\odot$$

For α Centauri B:

$$\mathcal{M}_B = 10^{\left(\frac{4.9 - M_{bolB}}{10.2}\right)}\mathcal{M}_\odot = 10^{\left(\frac{4.9 - 5.4757}{10.2}\right)}\mathcal{M}_\odot = 0.8781\mathcal{M}_\odot$$

(10%)

3 rd Iteration Repeating the step 1 until 4 and using the previous resulting mass in the second iteration:

Step 1 : $\mathcal{M}_A + \mathcal{M}_B = 1.1513 + 0.8781 = 2.0294$

Step 2 : $$p = \left[\frac{\alpha^3}{(\mathcal{M}_A + \mathcal{M}_B)P^2}\right]^{1/3} = \left[\frac{(17.59)^3}{(2.0294)(79.24)^2}\right]^{1/3} = 0.75314$$

Step 3 : For α Centauri A:

$$M_{bolA} = m_{bolA} + 5 + 5\log p = -0.11 + 5 + 5\log(0.7531) = 4.2751$$

For α Centauri B :

$$M_{bolB} = m_{bolB} + 5 + 5\log p = 1.09 + 5 + 5\log(0.7531) = 5.4751$$

Step 4 : For α Centauri A:

$$\mathcal{M}_A = 10^{\left(\frac{4.9 - M_{bolA}}{10.2}\right)}\mathcal{M}_\odot = 10^{\left(\frac{4.9 - 4.2751}{10.2}\right)}\mathcal{M}_\odot = 1.1515\mathcal{M}_\odot$$

For α Centauri B:

$$\mathcal{M}_B = 10^{\left(\frac{4.9 - M_{bolB}}{10.2}\right)}\mathcal{M}_\odot = 10^{\left(\frac{4.9 - 5.4751}{10.2}\right)}\mathcal{M}_\odot = 0.8782\mathcal{M}_\odot$$

(0%)

4 th Iteration Repeating the step 1 until 4 and using the previous resulting mass in the third iteration:

Step 1 : $\mathcal{M}_A + \mathcal{M}_B = 1.1515 + 0.8782 = 2.0297$

Step 2 : $$p = \left[\frac{\alpha^3}{(\mathcal{M}_A + \mathcal{M}_B)P^2}\right]^{1/3} = \left[\frac{(17.59)^3}{(2.0297)(79.24)^2}\right]^{1/3} = 0.7531$$

Step 3 : For α Centauri A:

$$M_{bolA} = m_{bolA} + 5 + 5\log p = -0.11 + 5 + 5\log(0.7531) = 4.2753$$

For α Centauri B :

$$M_{bolB} = m_{bolB} + 5 + 5\log p = 1.09 + 5 + 5\log(0.7531) = 5.4743$$

Step 4 : For α Centauri A:

$$\mathcal{M}_A = 10^{\left(\frac{4.9 - M_{bolA}}{10.2}\right)} \mathcal{M}_\odot = 10^{\left(\frac{4.9 - 4.2753}{10.2}\right)} \mathcal{M}_\odot = 1.1517 \mathcal{M}_\odot$$

For α Centauri B:

$$\mathcal{M}_B = 10^{\left(\frac{4.9 - M_{bolB}}{10.2}\right)} \mathcal{M}_\odot = 10^{\left(\frac{4.9 - 5.4743}{10.2}\right)} \mathcal{M}_\odot = 0.8784 \mathcal{M}_\odot$$

(0%)

5 th Iteration Repeating the step 1 until 4 and using the previous resulting mass in the fourth iteration:

Step 1 : $\mathcal{M}_A + \mathcal{M}_B = 1.1517 + 0.8784 = 2.0301$

Step 2 :

$$p = \left[\frac{\alpha^3}{(\mathcal{M}_A + \mathcal{M}_B)P^2}\right]^{1/3} = \left[\frac{(17.59)^3}{(2.0301)(79.24)^2}\right]^{1/3} = 0.7530''$$

Step 3 : For α Centauri A:

$$M_{bolA} = m_{bolA} + 5 + 5\log p = -0.11 + 5 + 5\log(0.7530) = 4.2740$$

For α Centauri B :

$$M_{bolB} = m_{bolB} + 5 + 5\log p = 1.10 + 5 + 5\log(0.7530) = 5.4740$$

Step 4 : For α Centauri A:

$$\mathcal{M}_A = 10^{\left(\frac{4.9 - M_{bolA}}{10.2}\right)} \mathcal{M}_\odot = 10^{\left(\frac{4.9 - 4.2740}{10.2}\right)} \mathcal{M}_\odot = 1.1518 \mathcal{M}_\odot$$

For α Centauri B:

$$\mathcal{M}_B = 10^{\left(\frac{4.9 - M_{bolB}}{10.2}\right)} \mathcal{M}_\odot = 10^{\left(\frac{4.9 - 5.4740}{10.2}\right)} \mathcal{M}_\odot = 0.8785 \mathcal{M}_\odot$$

(0%)

The results of the fifth iteration equal the results of the fourth iteration, so

The mass of α Centauri A : $\mathcal{M}_A = 1.1518\ \mathcal{M}_\odot$

The mass of α Centauri B : $\mathcal{M}_B = 0.8785\ \mathcal{M}_\odot$

III. The Age of Meteorite

1. From $N(t) = N_0 \exp(-\lambda t)$ and $D(t) = N_0 - N(t)$, then we have

$$D(t) = N(t)e^{\lambda t} - N(t) = N(t)(e^{\lambda t} - 1)$$
$$\frac{D(t)}{N(t)} = (e^{\lambda t} - 1) \text{ or } e^{\lambda t} = \frac{D(t)}{N(t)} + 1 \text{, so}$$
$$\lambda t = \ln\left(\frac{D(t)}{N(t)} + 1\right) \text{ and}$$
$$t = \frac{1}{\lambda}\ln\left(\frac{D(t)}{N(t)} + 1\right)$$

(10%)

2. Setting $N(t) = \frac{1}{2} N_0$, we get:

$$t_{1/2} = \ln 2/\lambda = 48.8 \times 10^9 \text{ year} = 48.8 \text{ Gyr}$$

(5%)

3. We know that ${}^{87}\mathrm{Rb} \rightarrow {}^{87}\mathrm{Sr}$, and taking the ratios $\frac{{}^{87}\mathrm{Sr}}{{}^{86}\mathrm{Sr}}$ as y (dependent variable) and $\frac{{}^{87}\mathrm{Rb}}{{}^{86}\mathrm{Sr}}$ as x (independent variable), we can use a simple linier regression equation:

$$y = a + bx$$

Taking into account the initial strontium, the corresponding equation is

$$\left(\frac{{}^{87}\mathrm{Sr}}{{}^{86}\mathrm{Sr}}\right) = \left(\frac{{}^{87}\mathrm{Sr}}{{}^{86}\mathrm{Sr}}\right)_0 + \left(e^{\lambda t} - 1\right)\left(\frac{{}^{87}\mathrm{Rb}}{{}^{86}\mathrm{Sr}}\right)$$

where **the intercept** $a = \left(\frac{{}^{87}\mathrm{Sr}}{{}^{86}\mathrm{Sr}}\right)_0$;

and **the slope** $b = \left(e^{\lambda t} - 1\right)$ **(20%)**

4. First, take the ratios:

Sample No	**Meteorite type**	**^{86}Sr (ppm)**	**^{87}Rb (ppm)**	**^{87}Sr (ppm)**	**^{87}Rb/^{86}Sr**	**^{87}Sr/^{86}Sr**
1	A	29.6	0.3	20.7	0.0101351	0.6993243
2	B	58.7	68.5	44.7	1.1669506	0.7614991
3	B	74.2	14.4	52.9	0.1940701	0.712938
4	A	40.2	7.0	28.6	0.1741294	0.7114428
5	A	19.7	0.4	13.8	0.0203046	0.7005076
6	B	37.9	31.6	28.4	0.8337731	0.7493404
7	A	33.4	4.0	23.6	0.1197605	0.7065868
8	B	29.8	105.0	26.4	3.5234899	0.885906
9	A	9.8	0.8	6.9	0.0816327	0.7040816
10	B	18.5	44.0	15.4	2.3783784	0.8324324

And we compile the data into Type A and B:

Type A

Sample No	**^{87}Rb/^{86}Sr (X)**	**^{87}Sr/^{86}Sr (Y)**	***XY***	***X^2***	***Y^2***
1	0.0101351	0.6993243	0.0070877	0.0001027	0.4890545
4	0.1741294	0.7114428	0.1238831	0.030321	0.5061509
5	0.0203046	0.7005076	0.0142235	0.0004123	0.4907109
7	0.1197605	0.7065868	0.0846212	0.0143426	0.4992649
9	0.0816327	0.7040816	0.0574761	0.0066639	0.4957309
Sum:	**0.4059623**	**3.5219431**	**0.2872916**	**0.0518425**	**2.4809121**

Λ	1.42E-11
$\overline{X}$	0.0811925
$\overline{Y}$	0.7043886
SS_{xx}	0.0188814
SS_{yy}	9.54601E-05
SS_{xy}	0.0013364
b_A	0.0707786
a_A	0.6986419

where,

SS_{xx} : sum of square for X = $\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2$

SS_{yy} : sum of square for Y = $\sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2$

SS_{xy} : sum of square for both X and $Y = \sum_{i=1}^{n} X_i Y_i - \frac{1}{n}\sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i$

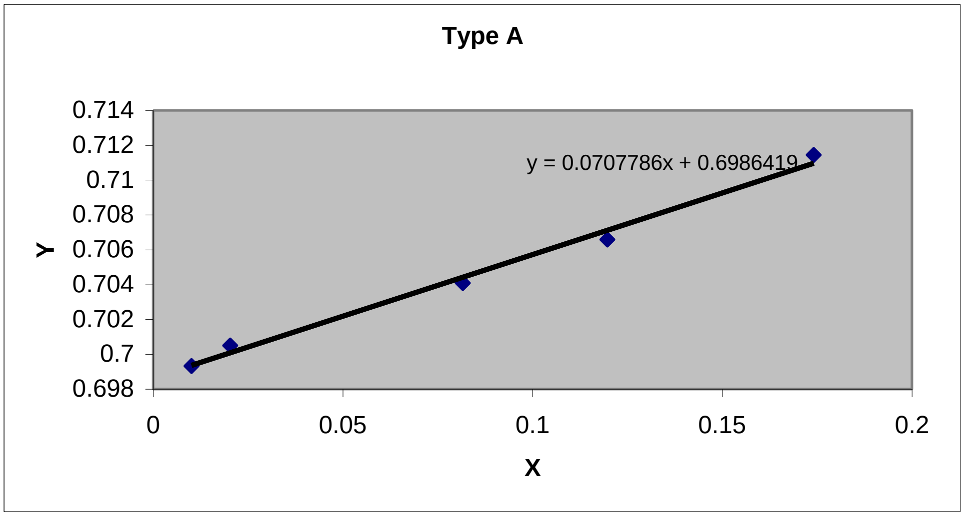
and **the slope**

$$b_A = \left(e^{\lambda t} - 1\right) = \frac{SS_{XY}}{SS_{XX}} = \frac{\sum_{i=1}^{5} X_i Y_i - \frac{1}{5}\sum_{i=1}^{5} X_i \sum_{i=1}^{5} Y_i}{\sum_{i=1}^{5} X_i^2 - \frac{1}{5}\left(\sum_{i=1}^{5} X_i\right)^2} = \frac{0.2872916 - 0.2859552}{0.0518425 - 0.0329611} = 0.0707786 \qquad (*)$$

and **the intercept**

$$a_A = \left(\frac{^{87}\mathrm{Sr}}{^{86}\mathrm{Sr}}\right)_0 = \bar{Y} - b\bar{X} = \overline{\left(\frac{^{87}\mathrm{Sr}}{^{86}\mathrm{Sr}}\right)} - b\overline{\left(\frac{^{87}\mathrm{Rb}}{^{86}\mathrm{Sr}}\right)}$$
$$= 0.7043886 - \ 0.0707786 \ \times 0.0811925 = 0.6986419$$

The scatter plot and linier regression line for Type A is:



(20%)

Type B

Sampel No	**^{87}Rb/^{86}Sr (X)**	**^{87}Sr/^{86}Sr (Y)**	***XY***	**X^2**	**Y^2**
2	1.1669506	0.7614991	0.8886318	1.3617737	0.5798809
3	0.1940701	0.7129380	0.1383599	0.0376632	0.5082806
6	0.8337731	0.7493404	0.6247799	0.6951776	0.561511
8	3.5234899	0.8859060	3.1214808	12.4149811	0.7848294
10	2.3783784	0.8324324	1.9798392	5.6566838	0.6929437
Sum:	**8.0966621**	**3.9421159**	**6.7530916**	**20.1662794**	**3.1274456**

λ	1.42E-11
$\bar{X}$	1.6193324
$\bar{Y}$	0.7884232
SS_{xx}	7.0550920
SS_{yy}	0.019390046
SS_{xy}	0.3694955
b_B	0.0523729
a_B	0.7036141

and **the slope** $b_B = \left(e^{\lambda t} - 1\right)$

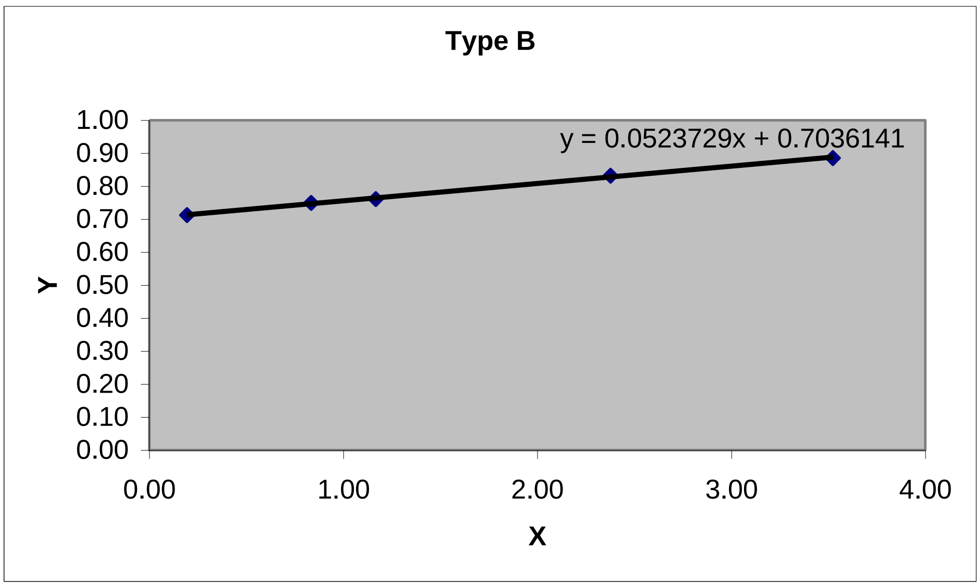
$$= \frac{SS_{XY}}{SS_{XX}} = \frac{\sum_{i=1}^{5} X_i Y_i - \frac{1}{5}\sum_{i=1}^{5} X_i \sum_{i=1}^{5} Y_i}{\sum_{i=1}^{5} X_i^2 - \frac{1}{5}\left(\sum_{i=1}^{5} X_i\right)^2} = \frac{6.7530916 - 6.3835961}{20.1662794 - 13.1111874} = 0.0523729$$

and **the intercept**

$$a_B = \left(\frac{{}^{87}\mathrm{Sr}}{{}^{86}\mathrm{Sr}}\right)_0 = \bar{Y} - b_B \bar{X} = \overline{\left(\frac{{}^{87}\mathrm{Sr}}{{}^{86}\mathrm{Sr}}\right)} - b_B \overline{\left(\frac{{}^{87}\mathrm{Rb}}{{}^{86}\mathrm{Sr}}\right)} = 0.7884232 - 0.0523729 \times 1.6193324$$

$=0.7036141$

The scatter plot and linier regression line for Type B is:



(20%)

5. Since we have 2 types of meteorites, then we have to calculate their ages separately, and λ = 1.42 × 10^{-11}. As $b = \left(e^{\lambda t} - 1\right)$, then $e^{\lambda t} = b+1$.
After we get a and b then

$$t = \frac{1}{\lambda}\ln\ \left(b+1\right)$$

which is 4.81592 × 10^9 year

From (*), we obtain

s_b	0.0040112
s_a	0.0009133
e_t	2.819138×10^{8}

sigma-b :

$$s_b = \sqrt{\frac{SS_{YY} - \frac{\left(SS_{XY}\right)^2}{SS_{XX}}}{\left(n-2\right)SS_{XX}}} = \sqrt{\frac{0.0000955 - \frac{\left(0.0013364\right)^2}{0.0188814}}{3\times0.0188814}} = 0.0040112$$

error on t : $e_t = \frac{1}{\lambda}\ln\left(s_b+1\right) = \frac{\ln\left(0.0040112+1\right)}{1.42\times10^{-11}} = 2.819138\times10^{8}$

sigma-a (error on initial) :

$$s_a = \sqrt{\frac{SS_{YY} - \frac{\left(SS_{XY}\right)^2}{SS_{XX}}}{\left(n-2\right)SS_{XX}}\times\sum_{i=1}^{n}X_i^2} = \sqrt{\frac{0.0000955 - \frac{\left(0.0013364\right)^2}{0.0188814}}{3\times0.0188814}\times0.0518425} = 0.0009133$$

Type B

Similar as in Type A, we obtain for Type B, in which $t = 3.595\times10^9$ year.

s_b	0.0013487
s_a	0.0060566
e_t	94915372.7530205

sigma-b :

$$s_b = \sqrt{\frac{SS_{YY} - \frac{\left(SS_{XY}\right)^2}{SS_{XX}}}{\left(n-2\right)SS_{XX}}} = \sqrt{\frac{0.0193900 - \frac{0.3694955^2}{7.0550920}}{3\times7.0550920}} = 0.0013479$$

error on t : $e_t = \frac{1}{\lambda}\ln\left(s_b+1\right) = \frac{\ln\left(0.0013479+1\right)}{1.42\times10^{-11}} = 9.48586\times10^{7}$

sigma-a (error on initial) :

$$s_a = \sqrt{\frac{SS_{YY} - \frac{\left(SS_{XY}\right)^2}{SS_{XX}}}{\left(n-2\right)SS_{XX}}\times\sum_{i=1}^{n}X_i^2} = \sqrt{\frac{0.0193900 - \frac{0.3694955^2}{7.0550920}}{3\times7.0550920}\times20.1662794} = 0.0060532$$

The Age of type A = (4.816 ± 0.282) × 10^9 year = 4.816 ± 0.282 Gyr

The Age of type B = (3.595 ± 0.095) × 10^9 year = 3.595 ± 0.435 Gyr

Hence, the type A is much older than the type B.

(20%)

6. The initial value of type A $(^{87}Sr/^{86}Sr)_0 = 0.6986419 \pm 0.0009133$
   The initial value of type B $(^{87}Sr/^{86}Sr)_0 = 0.7036141 \pm 0.0060532$ **(5%)**