Theoretical round. Sketches for solutions

Note for jury and team leaders. The proposed sketches are not full; the team leaders have to give more detailed explanations for students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

αβ-1. Dreams of the Polar Bear. It would seem that approaching 10 times closer to an object reduces its apparent magnitude by 5^m, and so approaching $1000 = 10^3$ times closer reduces the apparent magnitude by $3 \cdot 5^m = 15^m$. The magnitude of the Moon would be equal to $-12^m.7 - 15^m = -27^m.7$?

Would it? It turns out that in this case the Moon shines 2.3 times brighter than the Sun! In fact, this cannot happen!

Actually, the rule of reducing the magnitude by 5^m when approaching the object 10 times closer is valid only for point-like objects, or objects for which the small-angle approximation is acceptable, i.e., an approach 10 times closer to the object increases the apparent solid angle by a factor of 100. Indeed, the magnitude decrease of extended objects, like the lunar disc, is related to the increase of the object's apparent solid angle while keeping brightness unchanged.

The solid angle of the visible surface of the Moon from the Earth is

$$\Omega_1 = \pi \cdot (4{,}6 \cdot 10^{-3} \text{ rad})^2 \approx 6.7 \cdot 10^{-5} \text{ rad}^2.$$

When using the small-angle approximation, approaching 10^3 times closer would increase the visible area by the factor of 10^6 times, and it would become 67 rad^2. However, this solid angle cannot be larger than the solid angle of a hemisphere, 2π rad^2 = 6,3 rad^2. That is, in the calculations we have overestimated the area by a factor of at least 67 rad^2 / 6,3 rad^2 = 10.6, and make an error in the magnitude of at least $2^m.5 \cdot \log(10.6) = 2^m.6$. Thus, the magnitude of the lunar disk cannot be less than

$$-27^m.7 + 2^m.6 = -25^m.1.$$

More precise calculations show that when approaching 1000 times closer to the surface of the Moon (approximately from 380 thousand km to 380 km), the apparent solid angle will be

$$\Omega_2 = 2\pi \cdot (1 - \cos[\arcsin(R_M/[R_M + h])]) \text{ rad}^2 \approx 2\pi \cdot (1 - \cos[\arcsin(1.74/2.12)]) \text{ rad}^2 \approx 2{,}7 \text{ rad}^2,$$

i.e. it will increase by a factor of approximately $\Omega_2/\Omega_1 = 40\,000$ and (in this model of the same brightness of the lunar surface) the magnitude is

$$-12^m.7 - 2^m.5 \cdot \log(40000) \approx -24^m.2.$$

αβ-2. Great oppositions. Reducing the by 6.0% the semi-major axis of Mars orbit changes its sidereal period:

$$(T_1/T_0)^2 = (A_1/A_0)^3,$$

$$T_1 = T_0 \cdot (A_1/A_0)^{3/2} = 1.880 \text{ years} \cdot (0.94)^{3/2} = 1.713 \text{ years}.$$

Syndical period T_{1S} of new Mars may be found from formula

$$1/T_{1S} = 1/T_E - 1/T_1,$$

$$T_{1S} = T_1 T_E / (T_1 - T_E) = 2.402 \text{ years}.$$

That is, oppositions of Mars (of any kind) will occur every 2.402 years. After 5 such cycles, the Earth and Mars return approximately to the same positions. It is the new period for repeating the great oppositions. Answer is "every 12 years".

Note: Inaccurate data calculations (of T_1, for example) in this problem can easily lead to incorrect results. In calculating the synodic motion of Mars the exact value of the fractional part of the period is very important.

αβ-3. Proxima Centauri. Let us find the distance **d** between Proxima and Alpha Centauri.

$$\mathbf{d} = (x^2 + y^2 + z^2)^{1/2},$$

where x, y and z are projections of the distance on the axes of our coordinate system. Necessary data we should take from the "Data on some stars". Approximately

$$x = |\alpha_{PC} - \alpha_{\alpha C}| \times \cos\delta_{P\alpha C} \times L_{P\alpha C} = |\alpha_{PC} - \alpha_{\alpha C}| \times \cos\delta_{P\alpha C} / p_{P\alpha C},$$

$$y = |\delta_{PC} - \delta_{\alpha C}| \times L_{P\alpha C} = |\delta_{PC} - \delta_{\alpha C}| / p_{P\alpha C},$$

$$z = |L_{PC} - L_{\alpha C}| = |1/p_{PC} - 1/p_{\alpha C}|,$$

where α, δ, L, *p* are right ascension, declination, distance from the Earth, parallax, and values with sub-index PC are related to Proxima Centauri; those with sub-index αC, to Alpha Centauri; and PαC, to the middle point of Proxima and Alpha Centauri. So, from the "Data on some stars", we should take α, δ, and *p* of Proxima and Alpha Centauri. Calculations give us:

$$x = 0.0269 \text{ pc}, \quad y = 0.0425 \text{ pc}, \quad z = 0.0383 \text{ pc}, \quad \text{and so} \quad \mathbf{d} = 0.0632 \text{ pc}.$$

Thus, in comparison with observation from the Earth, the magnitude of Proxima Centauri while observed from the vicinity of Alpha Centauri is smaller by

$$\Delta m = 5^m \cdot \log(L_{P\alpha C}/\mathbf{d}) = -5^m \cdot \log(\mathbf{d} \cdot p_{P\alpha C}) \approx 6^m.57$$

than the magnitude while observed from the Earth.

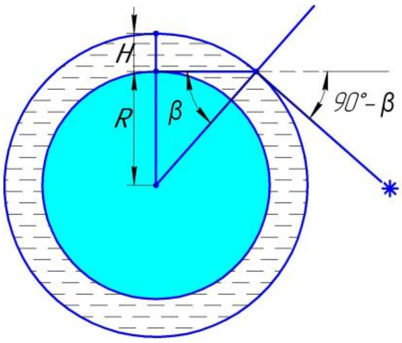
$$m = m_E - \Delta m = 11^m.05 - 6^m.57 \approx 4^m.5.$$

It is less than the limit value for naked eye, 6^m, so Proxima Centauri can be visible from the vicinity of Alpha Centauri by naked eye, **«да-yes»**.

Note: To prove the proposal ***«нет-no»*** *of the first question, it would not be necessary to make so many calculations of x, y, z and* ***d****. It is evident that* ***d*** *≥ z, so if Proxima Centauri cannot be visible from the distance z, it would be also not visible from the distance* ***d****.*

αβ-4. Hydroplanet. When passing the border between the airless space and water, rays are reflected:

$$\sin\alpha = n \cdot \sin\beta.$$



As a result, the rays go steeper after refraction. Due to the total reflection effect, only those rays reach observers at the outer surface of the ocean which are directed, in the water medium, so that their angle with respect to the normal to the water surface is below the critical value that occurs for sin α = 1, i.e. for α = 90°.

$$1 = n \cdot \sin\beta, \qquad \beta = \arcsin 1/n,$$

$$\beta = \arcsin 1/1.334 = \arcsin 0.7496 = 48°33'.$$

For low depth of the ocean, this is the largest zenith distance from which rays reach observers. When increasing the ocean's depth, this critical zenith angle increases because of the surface curvature. Drawing a plot, it is easy to find that light from the horizon line begins to come in the case of the minimal ocean depth that satisfies the condition:

$$\sin\beta = \boldsymbol{R}/(\boldsymbol{R}+\boldsymbol{H}), \qquad 1/n = \boldsymbol{R}/(\boldsymbol{R}+\boldsymbol{H}), \qquad \boldsymbol{H} = (n-1)\boldsymbol{R}.$$

4.1. Answer: $\boldsymbol{H} = 0.334\,\boldsymbol{R}$.

4.2. It appears from the same plot that the central star on the sky of the Hydroplanet passes, from its rise to its set, the angle

$$\tau = (90°-\beta) + 180° + (90°-\beta) = 360° - 2\beta = 262.12°.$$

In hour units, this is $17^h\,29^m$.

4.3. It appears from the same plot that it is the angle 90°-β that is analogous to the refraction angle at the horizon for such a planet.

$$\boldsymbol{r} = 90°-\beta = 41°27'.$$

α-5. Argali. Argali means the constellation Aries. The Sun is in this constellation in late April – early May, now it is almost in the opposite position on the ecliptic, so Aries is perfectly visible at night now. The brightest star in the constellation is Hamal, α Ari, its right ascension is $02^h\,07^m$. The upper culmination of a celestial body occurs when the sidereal time is equal to its right ascension. Sidereal time is equal to mean solar time at the date of autumnal equinox, and then begins to outstrip it with the speed of $1^d/365.2422$ per day (about $3^m\,56^s$ per day). 23 days have passed between the night of September 22/23 (autumnal equinox in 2014) and October 15-16 (the next 24 hours). The difference between mean solar time and sidereal time is now:

$$24^h\,00^m \times 23/365.2422 = 1^h\,31^m$$

So now sidereal time $02^h\,07^m$ corresponds to

$$02^h\,07^m - 1^h\,31^m = 0^h\,36^m \quad \text{of mean solar time.}$$

Cholpon-Ata is located in 13° to the West of the meridian 90° (the standard meridian for UT+6 Timezone), so the mean solar time here is behind the local time UT+6 used in Kyrgyzstan by $13° \times 4^{m/°} = 52^m$. So, to get the answer according to time UT+6 ,we have to add these 52^m, and we have:

$$0^h\,36^m + 0^h\,52^m = 1^h\,28^m \quad \text{of Kyrgyz time.}$$

So, tonight it will be a favorable period for the mountain sheep to observe the constellation of Aries.

The altitude of α Ari in upper culmination may be found using the formula:

$$h_U = 90° - \varphi + \delta = 90° - 42°39' + 23°28' = 70°49'.$$

Also we should calculate the altitude of α Ari in lower culmination:

$$h_L = 90° - \varphi + \delta = -90° + 42°39' + 23°28' = -23°53'.$$

β-5. Climate. In order to get an inner orbit of the star that would provide a seasonal cycle with a periodicity of 1 year, it is necessary that its close position with the Earth would take place once a year. That is, the synodic period of its revolution should be one year. It is not difficult to find that the sidereal period of revolution of the celestial body must be equal to

$$T = 1 \text{ year} \times 1 \text{ year} / (1 \text{ year} + 1 \text{ year}) = 0.5 \text{ years},$$

and the radius of its orbit, from the III Kepler's law:

$$R = R_0 \cdot (T_1/T_0)^{2/3} = 0.63\, R_0.$$

Thus, the new celestial body will once a year become close to the Earth at a distance of $R_0 - R = 0.37$ au, providing summer on the planet, and once a year move away to a distance of $R_0 + R = 1.63$ au, leaving the Earth in winter.
The ratio of energies received by the Earth from the new star in summer and winter will be

$$\ddot{E}_S/\ddot{E}_W = (R_0 + R)^2/(R_0 - R)^2 = 19.4.$$

The ratio of these energies to the energy $\ddot{E}_0$, received from the new star at a distance of 1 au is

$$\ddot{E}_S/\ddot{E}_0 = R_0^2/(R_0 - R)^2 = 7.3.$$

$$\ddot{E}_W/\ddot{E}_0 = R_0^2/(R_0 - R)^2 = 0.38.$$

At the same time, the Earth will always receive additional energy E_1 from the cooler Sun, that is, $E_S = E_1 + \ddot{E}_S$ in the summer, and $E_W = E_1 + \ddot{E}_W$. To keep the heat balance, all the incoming energy should be radiated.
The ratio of the energy radiated by the Earth in summer and winter will be

$$Э_S/Э_W = T_S^4/T_W^4 = (290\text{ K})^4/(270\text{ K})^4 = 1.33.$$

Neglecting the effects of delays of the seasons compared to the maxima and minima of the energy income, let us assume that $Э_S = E_S$, $Э_W = E_W$.
Then we can write:

$$E_1 + \ddot{E}_S = 1.33\cdot(E_1 + \ddot{E}_W),$$

$$E_1 + 19.4\cdot\ddot{E}_W = 1.33\cdot(E_1 + \ddot{E}_W),$$

$$(19.4 - 1.33)\cdot\ddot{E}_W = (1.33 - 1)\cdot E_1,$$

$$\ddot{E}_W / E_1 = 0.0183,$$

$$\ddot{E}_S / E_1 = 0.355.$$

The sum of E_1 and the average $\ddot{E}$ for the period must be the same as the Earth receives nowadays, E_0. To estimate the average value of $\ddot{E}$, let us take $\ddot{E}_0$, and so

$$\ddot{E}_0 / E_1 = 0.0486, \quad \ddot{E}_0 / E_0 = 0.0464, \quad E_1 / E_0 = 0.954.$$

5.1. Thus, the Sun in new conditions will emit 4.6% less energy, and so its temperature will be lower and equal to

$$T_1 = 0.954^{1/4}\cdot T_0 \approx 0.988\cdot T_0 \approx 5710 \text{ K}.$$

5.2. To understand what spectral type will correspond to the star, we calculate by how many magnitudes it is fainter than the Sun:

$$\Delta M = -5/2^m\cdot \lg(0.0464) \approx 3^m.3.$$

From the HR diagram, we find that it is about the spectral type K3.

α-6. Brightest stars. Due to precession of the Earth's axis, the North celestial pole moves in a circle around the North ecliptic pole, making one circle approximately every ~26000 years (the value can be clearly seen in the sky map of precession). The time till the CL century AD is approximately half of this period, and the North pole will change its position by approximately $2\varepsilon \approx 47°$ in the direction of the brightest star in Draco constellation, **γ** Dra or Etamin (in Draco the order of α-β-γ… is not the traditional one that starts from the brightest star, this fact is also clearly seen from the supplement materials). That is, in the CL century the visible part of the celestial sphere will be shifted about 47° in the direction to about the current 18^h RA.

Let us list the brightest stars (for example, twice more of them than the required four) and see whether they may be seen in Cholpon-Ata in the CL century. Now we can see stars with $\delta > -(90°-\varphi)+5° \approx -42°$ for any RA. After 13000 years the limit will move, $\delta > -42° + 47° = 5°$ for RA $\sim 6^h$ and $\delta > -42° - 47° = -89°$ for RA $\sim 18^h$ (according to current coordinates). The situation is more complicated for other right ascensions. Anyway, the center of invisible area of the sky will be at the point (6^h, -43°) and its radius will be $\rho = \varphi+5° = 48°$. By easy drawing we can find that

						Vis XXI	Vis CL
Сириус	Sirius	α CMa	$06^h\ 45^m\ 09^s$	-16° 42' 58"	$-1^m.46$	+	–
Канопус	Canopus	α Car	$06^h\ 23^m\ 57^s$	-52° 41' 45"	$-0^m.72$	–	–
Толиман	Toliman	α Cen	$14^h\ 39^m\ 36^s$	-60° 50' 07"	$-0^m.29$	–	+
Арктур	Arcturus	α Boo	$14^h\ 15^m\ 38^s$	19° 10' 57"	$-0^m.04^v$	+	+
Вега	Vega	α Lyr	$18^h\ 36^m\ 56^s$	38° 47' 01"	$0^m.03$	+	+
Капелла	Capella	α Aur	$05^h\ 16^m\ 41^s$	45° 59' 53"	$0^m.08$	+	+
Ригель	Rigel	β Ori	$05^h\ 14^m\ 32^s$	-08° 12' 06"	$0^m.12$	+	–
Процион	Procyon	α CMi	$07^h\ 39^m\ 18^s$	05° 13' 30"	$0^m.38$	+	–

That is, we will lose from the visible stars Sirius, Procyon and Rigel, and get **α** Centauri. The four brightest stars in the night sky of Cholpon-Ata in the CL century AD are: **α** Centauri, Arcturus, Vega and Capella.

β-6. Sombrero Galaxy.

6.1. All the lines in the spectra are shifted somewhat to longer wavelengths (redshifted). Obviously, M104 is receding from us due to the expansion of the Universe.

6.2. The overall receding velocity of M104 may be found from the redshift of the spectrum of the central region. Taking the average redshift of the four emission lines, we find that

$$z_{av} = (\Delta\lambda/\lambda)_{average} \approx 0.0035,$$

and since $z << 1$, the overall receding velocity of M104 is given by:

$$V = cz = 0.0036 \times 300000 \text{ km/s} = 1050 \text{ km/s}.$$

6.3. According to the Hubble law the distance to Sombrero Galaxy is

$$L = V/H = 1050 \text{ km/s} / 71 \text{ km/s/Mpc} \approx 15 \text{ Mpc}.$$

Note: In reality the distance to Sombrero Galaxy is 9 Mpc.
But we continue solution according to the calculated value, 15 Mpc.

The first and second spectra have different shifts than the third one. It indicates that the gas which is generating the emission lines has a motion relative to the center of M104. The amounts of these redshifts are not too clear but evidently less than the third one. To find the mass of the smaller region near the center, we will use the first spectrum as obtained closer to the center region. The distance from the center of the galaxy to this near-center region can be found by measuring sizes at the lower picture. The angular distance is approximately 0.05 of the width of the lower image, $\alpha = 0.05 \times 10.8'' = 0.54''$. To get the actual distance in a.u., it is easy to use the formula

$$R(\text{in a.u.}) = \alpha(\text{in arcsec}) \times L(\text{in pc}),$$

$$R(\text{a.u.}) = 0.54 \times 15 \cdot 10^{6} \approx 8 \cdot 10^{6}.$$

Note: In reality $R(\text{a.u.}) = 0.54 \times 9 \cdot 10^{6} \approx 5 \cdot 10^{6}$.

The average difference in redshifts of the first and third spectra may be estimated as about $\Delta \mathbf{z} \approx 0.0013$. Thus the velocity of the gas relative to the galactic center is about

$$V_1 = c\Delta \mathbf{z} = 0.0013 \times 300000 \text{ km/s} \approx 400 \text{ km/s}.$$

Then, let us assume that the gas in the region of the third spectrum revolves around the galactic center in a circular orbit with the period T. Its revolving velocity is

$$V_1 = 2\pi R/T = 400 \text{ km/s}.$$

By comparing with the average velocity of the Earth revolving around the Sun (A = 1 au, P = 1 year),

$$V_2 = 2\pi A/P = 30 \text{ km/s},$$

we find that

$$R/T = (V_1/V_2) \cdot (A/P) \approx 13 \text{ au/year}.$$

Assuming that the mass of the gas is much less than that of the galactic center and applying the Kepler's III law (in units $\mathbf{m}_\odot$ - au - year), we obtain the mass of the galactic center

$$M = R^3/T^2 \approx 13^2 \cdot R \approx 200 \times 8\ 10^{6} \approx 1.6 \cdot 10^{9} \text{ (in } \mathbf{m}_\odot).$$

Note: In reality, we should write $M \approx 200 \times 5 \cdot 10^{6}\ m_\odot \approx 10^{9}\ m_\odot$.

It is an approximation for the mass of the supermassive black hole in the center of M104.